

Open Systems Viewed Through Their Conservative Extensions

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A typical linear open system is often defined as a component of a larger conservative one. For instance, a dielectric medium, defined by its frequency dependent electric permittivity and magnetic permeability is a part of a conservative system which includes the matter with all its atomic complexity. A finite slab of a lattice array of coupled oscillators modelling a solid is another example. Assuming that such an open system is all one wants to observe, we ask how big a part of the original conservative system (possibly very complex) is relevant to the observations, or, in other words, how big a part of it is coupled to the open system? We study here the structure of the system coupling and its coupled and decoupled components, showing, in particular, that it is only the system's unique minimal extension that is relevant to its dynamics, and this extension often is tiny part of the original conservative system. We also give a scenario explaining why certain degrees of freedom of a solid do not contribute to its specific heat.

KEY WORDS: open systems, time-dispersive dissipative media, conservative systems, spectral theory

1. INTRODUCTION: OPEN SYSTEMS AND CONSERVATIVE EXTENSIONS

Our interest in open systems is motivated, as it often happens, by a few concrete problems which have something in common. One of the most important concerns a time-dispersive dissipative (TDD) dielectric medium and the fundamental problem of defining and studying the eigenmodes and, more generally, the spectral theory. Similar problems arise when considering an “open resonator” (or Helmholtz resonator), which is a regular resonator coupled to an exterior system

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with an absolutely continuous spectrum. A third problem originates in statistical mechanics, when one considers a finite cube or slab as a small part of an ideal solid, modeled by a lattice array of coupled oscillators, and wonders how much information can be extracted about the entire solid from observations made only from within the finite part. The common feature of above dynamical systems is that they are open, in other words, the time dynamics do not preserve the energy and/or the material but exchange them with an exterior which often is not observable. It turns out that it is possible to reach some interesting conclusions about properties of open systems based on their minimal conservative extensions introduced recently in Ref. 2. This is the subject of this paper.

As an indication of some of what lies ahead, we mention a “toy” example of a mechanical system, which we discuss in Sec 3. The example shows how a high degree of symmetry in the system, which is related to high spectral multiplicity of the governing operator, results in many motions of the system being unaffected by a coupling to another system of “hidden” variables. Indeed, comparisons of the computation of the specific heat of a crystalline solid to experiment indicate that certain motions, or degrees of freedom, of the structure have to be left out—they are “frozen” (Ref. 4, Sec. 3.1, Ref. 5, Sec. 6.4). One of our main theorems (Theorem 15) applies to this type of situation—it describes how the multiplicity of the modes of a system that are affected by a coupling to another system is bounded by the rank of the coupling. Another example of a possible application of this work is one that inspired us to begin the study, although we do not pursue it at this point. It concerns a periodic dielectric waveguide that admits “nonrobust” modes at certain isolated wave number and frequency pairs. These are true (nonleaky) modes that become leaky through radiation loss under perturbation of the frequency or wavenumber. The system of modes in the waveguide and the exterior system, characterized by extended states in the surrounding air, become coupled, and this coupling produces interesting transmission anomalies.⁽¹⁰⁾

One can think of two intimately related and complementing ways to define an open system: (i) *intrinsic description by a non-conservative evolution equation*; (ii) *as a subsystem of a conservative system*. Taking the intrinsic description as basic we define an *open system* as one governed by a causal time-homogeneous linear evolution equation

$$m\partial_t v(t) = -iAv(t) - \int_0^\infty a(\tau)v(t - \tau) d\tau + f(t), \quad v(t) \in H_1, \quad (1)$$

in which H_1 is a separable Hilbert space, m and A are self-adjoint operators in H_1 with $m > 0$, and $f(t)$ is an external force in H_1 . We always assume that *the system is at rest for negative times* $t \leq 0$, in other words the following *rest condition* is satisfied

$$v(t) = 0, \quad f(t) = 0 \quad \text{for } t < 0. \quad (2)$$

The integral term in (1) involving the operator-valued response function $a(\tau)$ is a subject to the *dissipation (no-gain) condition*

$$\operatorname{Re} \int_0^\infty \int_0^\infty \overline{v(t)} a(\tau) v(t - \tau) dt d\tau \geq 0 \quad \text{for all } v(t) \text{ with compact support.} \quad (3)$$

Evidently, the integral term in (1) is responsible for the non-conservative, or open, nature of the system. Its form explicitly accounts for the system’s causality and time-homogeneity. The friction function $a(t)$ represents both delayed response and instantaneous friction; thus we take it to be of the form

$$a(t) = a_\infty \delta(t) + \alpha(t), \quad (4)$$

where the coefficient of *instantaneous friction* a_∞ is a bounded non-negative operator in H_1 and the *delayed response function* $\alpha(t)$ is strongly continuous and bounded as an operator-valued function of t with respect the norm in $\mathcal{B}(H_1)$, the space of bounded operators in H_1 .

The other important view on an *open system is that it is a subsystem* of a given conservative (conservative) system $(\mathcal{H}, \mathcal{A})$, described by conservative evolution equation

$$\mathcal{M} \partial_t \mathcal{V}(t) = -i\mathcal{A}\mathcal{V}(t) + \mathcal{F}(t), \quad \mathcal{V}, \mathcal{F} \in \mathcal{H}, \quad (5)$$

where \mathcal{H} is a separable Hilbert space, \mathcal{A} is a self-adjoint operator in it, and $\mathcal{F}(t)$ is an external force, with a subsystem identified by a subspace $H_1 \subset \mathcal{H}$. We refer to the subsystem’s space H_1 as the *observable variables*.

An intimate relation between the two ways of looking at open systems can be described as follows, Refs. 2, 3:

- (i) an open system defined by (1) and satisfying (3) can always be represented as a subsystem of a conservative extension in the form (5), and, *if minimal, such an extension is unique* up to isomorphism;
- (ii) the evolution of a subsystem of a conservative system (5) can be represented in the form (1) with a friction function $a(t)$ satisfying (3).

More precisely, taking the subsystem point of view on an open system we can identify an open system (1) with a *subsystem of its minimal conservative extension* $(\mathcal{H}, \mathcal{A})$ in which a subspace $H_1 \subset \mathcal{H}$ acts as the space of observable variables. Then we define the open system’s exterior as the orthogonal complement $H_2 = \mathcal{H} \ominus H_1$, referring to it as the *hidden variables*. Having the decomposition $\mathcal{H} = H_1 \oplus H_2$ we can recast the evolution Eq. (5) into the following system (see Ref. 2, Sec. 2).

$$\begin{aligned} m_1 \partial_t v_1(t) &= -iAv_1(t) - i\Gamma v_2(t) + f_1(t), \quad m_1 > 0, A \text{ is self-adjoint,} \quad (6) \\ \partial_t v_2(t) &= -i\Gamma^\dagger v_1(t) - i\Omega_2 v_2(t), \Omega_2 \text{ is self-adjoint,} \end{aligned}$$

where $v_1 \in H_1$, $v_2 \in H_2$, and $\Gamma : H_2 \rightarrow H_1$ is the *coupling operator*. By its very form, the system (6), involving the coupling operator Γ and its adjoint $\Gamma^\dagger : H_1 \rightarrow H_2$, is explicitly conservative, and Γ^\dagger and Γ determine the channels of “communication” from the observable to the hidden and back from the hidden to the observable. Observe that if one solves the second equation in (6) for $v_2(t)$ and inserts it into the first equation, the resulting equation will be of the form (1) with friction function

$$a(t) = \Gamma e^{-i\Omega_2 t} \Gamma^\dagger, \quad t \geq 0, \quad (7)$$

which, as is easy to verify, always satisfies the dissipation condition (3).

In a typical example of an open system embedded within a given conservative system, this conservative system is not necessarily minimal as a conservative extension of the open system. The minimal conservative extension is often much simpler system than the original one. For instance, a time dispersive and dissipative (TDD) dielectric medium, as described by the Maxwell equations with frequency dependent electric permittivity ε and magnetic permeability μ , constitutes an open system. Note that such ε and μ arise through the interaction of the electromagnetic fields with the molecular structure of the matter, which plays the part of the hidden variables. But if, however, ε and μ are all that is known, clearly these functions would not allow one to *reconstruct* the full molecular structure of the matter but rather only its minimal conservative extension.

Another simple but instructive example is provided by a general *scalar (one-dimensional) open system* as described by (1) with $H_1 = \mathbb{C}$ and friction function $a(t)$ satisfying (3) and (4). Observe that the classical damped oscillator with $a(t) = a_\infty \delta(t)$ is a particular case of such general scalar open system. The minimal conservative extension of a general scalar open system is described by a triplet $\{H_2, \Omega_2, \Gamma\}$ such that (7) holds and its elements H_2 , Ω_2 and Γ are constructed as follows, (Ref. 2, Sec. 4.1, 5.1, 5.2, A. 2). First, using the Bochner Theorem, we obtain the following representation of the friction function

$$a(t) = \int_{-\infty}^{\infty} e^{-i\omega t} dN(\omega) \quad (8)$$

with a unique, non-decreasing, right-continuous bounded function $N(\omega)$ defining a nonnegative measure $N(d\omega)$ on the real line \mathbb{R} . Then

$$H_2 = L^2(\mathbb{R}, N(d\omega)), \quad (9)$$

the operator Ω_2 is the multiplication by ω on $L^2(\mathbb{R}, N(d\omega))$, i.e.

$$[\Omega_2 \psi](\omega) = \omega \psi(\omega), \quad \omega \in \mathbb{R}, \quad \psi \in L^2(\mathbb{R}, N(d\omega)), \quad (10)$$

and the coupling operator Γ and its adjoint are

$$\Gamma[\psi(\cdot)] = \int_{-\infty}^{\infty} \psi(\omega)N(d\omega) : L^2(\mathbb{R}, dN) \rightarrow \mathbb{C}, \tag{11}$$

$$[\Gamma^\dagger v](\omega) = v, \quad v \in \mathbb{C}, \quad \omega \in \mathbb{R}.$$

Consequently, the minimal conservative extension of the form (6) becomes here

$$m \partial_t v = -iAv - i \int_{-\infty}^{\infty} \psi(\omega)d\omega + f(t), \tag{12}$$

$$\partial_t \psi(\omega) = -iv - i\omega\psi(\omega), \quad \psi \in L^2(\mathbb{R}, N(d\omega)).$$

In the case of the classical damped oscillator the measure $N(d\omega)$ is just the Lebesgue measure, i.e. $N(d\omega) = d\omega$, and the system (12) is equivalent to the Lamb model (see Refs. 6 and 13), which is a point mass attached to a classical elastic string (with ω being the wave number). In the case of a general spectral measure $N(d\omega)$ one can view the minimal extension (12) as one obtained by attaching a point mass m to a general “string” as described by a simple, i.e., multiplicity-one, self-adjoint operator with the spectral measure $N(d\omega)$. This point of view is justified by a *fundamental construction due to M. G. Krein of a unique “real” string corresponding to any given spectral measure*. This construction as a part of an exhaustive study of relations between the spectral measure, the corresponding admittance operator (the coefficient of dynamical compliance), and strings, is presented in two papers^(11,12) by I. S. Kac and M. G. Krein.

To clarify the exact meaning of a string, we give a brief description of a loaded string S_1 $[0, L]$ on an interval $[0, L]$, $0 \leq L \leq \infty$, as it is presented by Kac and Krein^(11,12). We assume (i) the string S_1 $[0, L]$ has constant stiffness 1; (ii) a non-decreasing nonnegative function $M(s)$, $s \geq 0$, describes its mass distribution, with $M(s)$ being the total string mass on the interval $[0, s]$. The string states are complex-valued functions $\psi(s)$, $0 \leq s \leq L$, from the Hilbert space $L^2([0, L], M(ds))$. The string dynamics is governed by the following, equation

$$\frac{\partial^2 \psi}{\partial t^2}(s, t) = A_M[\psi](s, t), \quad 0 \leq s \leq L, \tag{13}$$

where the string operator A_M is defined by the expression

$$A_M[\psi](s) = -\frac{d}{dM(s)} \frac{d\psi}{ds}(s), \quad 0 \leq s \leq L, \tag{14}$$

with the boundary conditions

$$\psi'(0) = 0, \quad \psi'(L)h + \psi(L) = 0, \quad \text{where } h \text{ is real.} \tag{15}$$

We do not formulate the original statements from (Ref. 12, Theorem 11.1, 11.2), because of the considerable space needed to introduce and define all relevant

concepts, but their principal point is that any nonnegative measure $N(d\omega)$ on $(-0, \infty)$ satisfying the condition

$$\int_{-0}^{\infty} \frac{N(d\omega)}{1 + \omega} < \infty \quad (16)$$

is the spectral measure of a unique string as described by the self-adjoint operator A_M defined by (Refs. 14,15). In another words, given a nonnegative measure $N(d\omega)$ on the positive semiaxis $(-0, \infty)$ which satisfies the condition (16), one can construct a unique mass distribution $M(s)$, so that the corresponding string $S_1[0, L]$ has $N(d\omega)$ as its spectral measure. As to the relation between $N(d\omega)$ and $M(s)$ a number of insightful examples are provided in (Ref. 12, Sec. 11–13).

Observe now that, if a scalar open system is described by (6), then *regardless of how complex the original triplet $\{H_2, \Omega_2, \Gamma\}$ is, its minimal counterpart $\{H_{2,\min}, \Omega_{2,\min}, \Gamma_{\min}\}$ is always of the universal form (12), and one can think of it as obtained by attaching a string to a point mass.* The concept of a string, as represented by the spectral measure $N(d\omega)$ on \mathbb{R} , turns out to be useful in describing the minimal extension of a multidimensional open system, where one has to use a number of strings for its construction. In particular, we will introduce a rather simple *string spectral decomposition* for an arbitrary self-adjoint operator Ω_2 for which the number of strings equals exactly to the spectral multiplicity of Ω_2 , and *a single string has always spectral multiplicity one.* We use then the number of strings involved in the string decompositions to characterize their relative complexity.

We reiterate the important observation that follows from the above examples and discussion: *the evolution of a subsystem is fully described by its minimal conservative extension similar to the system (12) which, typically, is substantially simpler than the original conservative system.* Consequently, a significant part of the modes of the original system can be completely decoupled from the open system. These observations make the minimal conservative extension an attractive instrument: (i) it is a simpler substitute for often enormously complex original conservative systems (as the atomic structure of the matter), (ii) since it is conservative, the classical spectral theory is available, and (iii) it provides information about how much of the original system is reconstructible by an observer in the open subsystem. Based on our considerations hitherto, we see our objectives as follows:

- (i) identify information about a conservative system that is carried by its subsystem (*reconstructibility*);
- (ii) relate the unique minimal extension of an open system to a given “original” larger conservative system;
- (iii) study the coupling operator of a subsystem and to identify which part of the system is coupled through it.

- (iv) understand the decomposition of open systems through simultaneous decompositions of the internal dynamics of the observable and hidden variables and the coupling operator between them.

Organization of the paper. Section 2 gives precise definitions and concise mathematical discussions of the concepts introduced so far, which will serve as background for the development of the work.

In Sec. 3, we construct a toy model of a solid that has frozen degrees of freedom. This example serves to illustrate the role that a high degree of system symmetry plays in decoupling parts of a dynamical system, as we have already discussed (p. 2).

Section 4 concerns the *reconstructibility* of conservative systems from open systems, in particular, from the dynamics projected to the observable and to the hidden state variables. The section culminates in one of our main theorems, Theorem 15, which describes how the number of coupling channels between the observable and hidden variables bounds the number of strings required in the construction of the minimal extension.

In Sec. 5, we investigate the decomposition, or decoupling, of open systems by means of the minimal conservative extension. We show first the equivalence between (i) the decoupling of the dynamics in a subspace of the open system from the dynamics in the complementary part of the open system, which is determined by A_1 and $a(t)$, and (ii) splittings of the conservative extension that are invariant under A_1 , Ω_2 , and Γ , or, equivalently, that are preserved in \mathcal{H} by Ω and the projection operator to H_1 . We then make an analysis of the relation between splittings of the conservative extension \mathcal{H} that preserve its dynamics (equivalently, splittings of the projected open system on H_1) and the singular-value decomposition of the coupling operator Γ . If the friction function involves no instantaneous friction component, that is, if $a_\infty = 0$, then the coupling Γ is bounded; otherwise it will be unbounded. More general assumption when Γ is bounded with respect to the frequency operator Ω_2 , which covers the case of nonzero instantaneous friction a_∞ , is considered in (Ref. 2, Sec. 2.2). In this work we focus on the case of bounded coupling, in which the analysis is more transparent.

To preserve the conceptual transparency of the arguments and results and to make them more readily accessible to the reader, we forgo full rigorous arguments in the development of the ideas and present more elaborate statements of the theorems as well as their proofs in Sec. 6.

2. CONSERVATIVE EXTENSION AND SPECTRAL COMPOSITION

Our study of the general open linear DD system is based on the ability to embed it in a unique way into a larger conservative system, in which the observable system is complemented by a space of hidden degrees of freedom. The frequency

operator for the hidden variables gives rise to a (nonunique) decomposition of these variables into subspaces that are interpreted as independent “strings” that are “attached” to the system of observable variables and account for the dissipative and dispersive effects that cause this system to be open. Each string is characterized by a spectral measure, and exactly how they strings are attached to the observable variables is described by the coupling operator. In this section, we give the background for constructing this conservative extension and discuss the spectral composition of the space of hidden variables into strings and the structure of the coupling operator.

We reiterate our definition of an open linear DD system and the conditions it satisfies. We take an open system to be of the form

$$m\partial_t v(t) = -iAv(t) - \int_0^\infty a(\tau)v(t - \tau) d\tau + f(t), \quad v(t) \in H_1, \quad (17)$$

in which H_1 is a separable Hilbert space, m and A are self-adjoint operators in H_1 with $m > 0$, and $f(t)$ is an external force in H_1 . The function $a(t)$ is subject the dissipation (no-gain) condition

Condition 1. (dissipation) Let $a(t) = a_\infty\delta(t) + \alpha(t)$, where a_∞ is a bounded non-negative operator in H_1 and $\alpha(t)$ is a strongly continuous and bounded operator-valued function of t with respect the operator norm in $\mathcal{B}(H_1)$. $a(t)$ satisfies the dissipation condition if

$$\operatorname{Re} \int_0^\infty \int_{-\infty}^\infty \overline{v(t)}a(\tau)v(t - \tau) dt d\tau \geq 0 \quad \text{for all } v(t) \text{ with compact support.} \quad (18)$$

The systems we consider will satisfy the rest condition

Condition 2. (rest condition) An open system satisfies the rest condition if, for all $t < 0$, $f(t) = 0$ and $v(t) = 0$.

Under the assumption of the rest condition, the inner integral in equation (18) can be taken from zero to infinity.

The minimal extension. The following statement, which is a generalization of the Bochner theorem, plays the key role in the embedding of the open system (1) into a unique minimal conservative extension (6) (Ref. 2, Theorem 3.2). Given that the dissipation condition (3) is satisfied, the Proposition 3 provides the existence of the space H_2 of hidden variables, the frequency operator Ω_2 for its internal dynamics, and the coupling operator Γ .

Proposition 3. (minimal extension) Let $\mathcal{B}(H_1)$ be the space of all bounded linear operators in H_1 . Then a strongly continuous $\mathcal{B}(H_1)$ -valued function $a(t)$, $0 \leq t < \infty$, is representable as

$$a(t) = \Gamma e^{-it\Omega_2} \Gamma^\dagger, \tag{19}$$

with Ω_2 a self-adjoint operator in a Hilbert space H_2 and $\Gamma : H_1 \rightarrow H_2$ a bounded linear map, if and only if $a(t)$ satisfies the dissipation condition (3) for every continuous H_1 valued function $v(t)$ with compact support. If the space H_2 is minimal—in the sense that the linear span

$$\langle f(\Omega_2)\Gamma^\dagger v : f \in C_c(\mathbb{R}), v \in H_1 \rangle \tag{20}$$

is dense in H_2 — then the triplet $\{H_2, \Omega_2, \Gamma\}$ is determined uniquely up to an isomorphism.

Remark 4. In fact, it is sufficient to assume that $a(t)$ is locally bounded and strongly measurable, strong continuity then follows from (19).

Since, as it turns out, spans similar to (20) arise often in the analysis of open systems, we name them closed orbits and define them as follows.

Definition 5. (orbit) Let Ω be a self-adjoint operator in a Hilbert space H and S is a subset of vectors in H . Then we define the closed orbit (or simply orbit) $\mathcal{O}_\Omega(S)$ of S under action of Ω by

$$\mathcal{O}_\Omega(S) = \text{closure of span } \{f(\Omega)w : f \in C_c(\mathbb{R}), w \in S\}. \tag{21}$$

If H' is a subspace of H such that $\mathcal{O}_\Omega(H') = H'$, then H' is said to be invariant with respect to Ω or simply Ω -invariant.

If Ω is bounded, the orbit $\mathcal{O}_\Omega(S)$ is equal to the smallest subspace of H containing S that is invariant, or closed, under Ω . Equivalently, it is the smallest subspace of H containing S that is invariant under $(\Omega - i)^{-1}$; this latter formulation is also valid for unbounded operators. The relevant theory can be found, for example, in Ref. 1 or 7. The orbit of S under the of two self-adjoint operators Ω and A can be defined by application of continuous functions of Ω and A to elements of S , but we shall only need the characterization that

$$\begin{aligned} \mathcal{O}_{\Omega,A}(S) \text{ is the smallest subspace of } H \text{ containing } S \\ \text{that is invariant under } (\Omega - i)^{-1} \text{ and } (A - i)^{-1}. \end{aligned} \tag{22}$$

Proposition 3 allows one uniquely to construct the triple $\{H_2, \Omega_2, \Gamma\}$ and, consequently, the minimal conservative extension based on the observable friction function $a(t)$. In fact, there is a statement similar to Proposition 3 which holds for

$a(t)$ of the most general form (4), in which instantaneous friction is included; see (Ref. 2, Theorem 7.1). Consideration of the instantaneous friction term leads to an unbounded coupling operator Γ ; the treatment of unbounded coupling is technical, and we do not consider it in this work, but treat it in a forthcoming exposition.

It is worthwhile to understand the idea behind the construction of the triple $\{H_2, \Omega_2, \Gamma\}$, as it shows plainly how the time-harmonic decomposition of $a(t)$ determines the spectral structure of H_2 . We therefore take a page to explain it. Introduce the Fourier-Laplace transform of $a(t)$:

$$\hat{a}(\zeta) = \int_0^\infty a(t)e^{i\zeta t} dt, \quad \text{for } \Im \zeta > 0. \quad (23)$$

It turns out that the dissipation condition, Condition 1, on $a(t)$ is equivalent to the condition that $\hat{a}(\zeta)$ is a Nevanlinna function: it is an analytic function of the open upper half plane with values that have positive real (self-adjoint) part. The restriction of the real part of $\hat{a}(\zeta)$ to the real line ($\hat{a}(\omega)$ for $\omega \in \mathbb{R}$ is the Fourier transform of $a(t)$) is no longer a classical function in general, but rather a *nonnegative* operator-valued measure $dN(\omega)$. One then takes H_2 to be the space of square-integrable functions from \mathbb{R} to H_1 with respect to this measure:

$$H_2 = L^2(\mathbb{R}, H_1, dN(\omega)), \quad (24)$$

for which the inner product is defined by

$$\langle f|g \rangle_{H_2} = \frac{1}{\pi} \int_{\mathbb{R}} \langle f(\omega)|dN(\omega)g(\omega) \rangle_{H_1}, \quad (25)$$

with the integral understood in the Lebesgue–Stieltjes sense. The operator Ω_2 is simply multiplication by ω :

$$(\Omega_2(g))(\omega) = \omega g(\omega) \quad \text{for } g \in H_2, \quad (26)$$

the adjoint $\Gamma^\dagger : H_1 \rightarrow H_2$ of the coupling operator is defined by sending $v \in H_1$ to the function with constant value v :

$$(\Gamma^\dagger(v))(\omega) = v \quad \text{for all } \omega \in \mathbb{R} \quad (27)$$

and $\Gamma : H_2 \rightarrow H_1$ is given by

$$\Gamma(f) = \frac{1}{\pi} \int_{\mathbb{R}} f(\omega) dN(\omega). \quad (28)$$

One can check that this extension indeed produces the friction function $a(t)$ by observing that, since $\hat{a}(\zeta)$ is a Nevanlinna function, it is constructible from $dN(\omega)$ by the Cauchy transform:

$$i\hat{a}(\zeta) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\zeta - \omega} dN(\omega), \quad (29)$$

and when applied to a vector $v \in H_1$, gives

$$i\hat{a}(\zeta)v = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\zeta - \Omega_2} (\Gamma^\dagger(v)(\omega)) dN(\omega) = \left(\Gamma \frac{1}{\zeta - \Omega_2} \Gamma^\dagger \right) v, \quad (30)$$

which is the Fourier-Laplace transform of $-i(\Gamma e^{-i\Omega_2 t} \Gamma^\dagger)v$.

The coupling channels. Now let us understand the structure of the coupling operator Γ well. There is a canonical isomorphism between the ranges of Γ and Γ^\dagger :

$$U : \text{Ran } \Gamma \rightarrow \text{Ran } \Gamma^\dagger. \quad (31)$$

This isomorphism is constructed as follows: Observe that

$$H_1 = \overline{\text{Ran } \Gamma} \oplus \text{Null } \Gamma^\dagger \quad \text{and} \quad H_2 = \overline{\text{Ran } \Gamma^\dagger} \oplus \text{Null } \Gamma \quad (32)$$

so that Γ and Γ^\dagger are determined by their actions on $\text{Ran } \Gamma^\dagger$ and $\text{Ran } \Gamma$, respectively. Denote their restrictions to these subspaces (both in domain and target space) by

$$\Gamma_R = \Gamma \upharpoonright \text{Ran}(\Gamma^\dagger) \quad \text{and} \quad \Gamma_R^\dagger = \Gamma^\dagger \upharpoonright \text{Ran}(\Gamma) = (\Gamma_R)^\dagger. \quad (33)$$

U is then given explicitly by

$$U = (\Gamma_R^\dagger \Gamma_R)^{-1/2} \Gamma_R^\dagger = \Gamma_R^\dagger (\Gamma_R \Gamma_R^\dagger)^{-1/2}. \quad (34)$$

The positive operators $(\Gamma_R \Gamma_R^\dagger)^{1/2}$ on $\text{Ran } \Gamma$ and $(\Gamma_R^\dagger \Gamma_R)^{1/2}$ on $\text{Ran } \Gamma^\dagger$ have trivial nullspace and are related through U by³

$$\Gamma_R^\dagger = U(\Gamma_R \Gamma_R^\dagger)^{1/2} = (\Gamma_R^\dagger \Gamma_R)^{1/2} U. \quad (35)$$

This is the polar decomposition of Γ^\dagger . It allows one to define the “coupling channels” in a natural way as the pairing of the eigenmodes of the positive part $(\Gamma_R \Gamma_R^\dagger)^{1/2}$ in $\text{Ran } \Gamma^\dagger \subset H_1$ with the corresponding eigenmodes of $(\Gamma_R^\dagger \Gamma_R)^{1/2}$ in $\text{Ran } \Gamma \subset H_2$ through U . Γ^\dagger and Γ provide a direct coupling between these modes—hence the term “coupling channel.” In the case of unbounded coupling or continuous spectrum, the modes are not genuine vectors, but are members of an appropriate furnishings of H_1 and H_2 . For continuous spectrum, we may also define coupling channels more generally as pairs of spaces identified through U that are fixed by the positive operators in (35) and therefore mapped to one another

³ It is not always necessary to deal with the restrictions Γ_R or Γ_R^\dagger ; often Γ or Γ^\dagger itself is suitable. For example, $\Gamma \Gamma^\dagger$ and $\Gamma_R \Gamma_R^\dagger$ coincide on the domain of the latter, and the former maps the orthogonal complement of this domain to zero. The analogous statement holds for $\Gamma^\dagger \Gamma$ and $\Gamma_R^\dagger \Gamma_R$. $U^{-1} : \text{Ran } \Gamma^\dagger \rightarrow \text{Ran } \Gamma$ is given by $U^{-1} = U^\dagger = (\Gamma_R \Gamma_R^\dagger)^{-1/2} \Gamma_R = \Gamma_R (\Gamma_R^\dagger \Gamma_R)^{-1/2}$, and Γ has the polar decompositions $\Gamma_R = U^{-1} (\Gamma_R^\dagger \Gamma_R)^{1/2} = (\Gamma_R \Gamma_R^\dagger)^{1/2} U^{-1}$.

by Γ^\dagger and Γ . We use this structure amply in Sec. 5, which deals with decomposition of open systems.

Definition 6. (coupling channel) *A coupling channel is a pair (S_1, S_2) , in which S_1 is an invariant space of $(\Gamma_R \Gamma_R^\dagger)^{1/2}$ in $\text{Ran } \Gamma \subset H_1$ and S_2 is an invariant space of $(\Gamma_R^\dagger \Gamma_R)^{1/2}$ in $\text{Ran } \Gamma^\dagger \subset H_2$ such that $U(S_1) = S_2$ (see (31) and (34) for the definition of U). It follows that $\Gamma^\dagger(S_1) = S_2$ and $\Gamma(S_2) = S_1$. A simple coupling channel is a coupling channel in which the members S_1 and S_2 are one-dimensional. A simple coupling channel corresponds to a pair of eigenmodes (ϕ_1, ϕ_2) of $\Gamma \Gamma^\dagger$ and $\Gamma^\dagger \Gamma$ for the same eigenvalue.*

The extending strings. The conservative system (H_2, Ω_2) , consisting of the space of hidden variables together with its operator of internal dynamics, can be interpreted as a set of independent abstract “strings” to which the system (H_1, m, A) is attached by the coupling channels defined by Γ . The following spectral decomposition of H_2 with respect to Ω_2 is obtained by a straightforward modification of Theorem VII. 6 in Ref. 7:

$$H_2 \cong \bigoplus_{j=1}^M L^2(\mathbb{R}, \mathbb{C}, d\mu_j(\omega)), \quad d\mu_{j+1} \leq d\mu_j, \tag{36}$$

in which “ \leq ” denotes absolute continuity of measures and Ω is represented by multiplication by the independent variable ω . We call each component of this decomposition a “string”; the j -th string is generated by a function $f_j(\omega)$ of maximal spectral type in $L^2(\mathbb{R}, \mathbb{C}, d\mu_j(\omega))$, that is, $f_j(\omega) \neq 0$ almost everywhere with respect to $d\mu_j$. A string is characterized by its invariance under the action of Ω_2 and by the property that the restriction of Ω to the string has multiplicity 1. Of course, the measures μ_j need not be taken to be nested by absolute continuity; even if they are, a decomposition into strings is not unique. The strings are decoupled from each other with respect to the action of Ω_2 , that is, within the conservative system (H_2, Ω_2) .

Definition 7. (string) *An abstract string, or simply a string in the system (H_2, Ω_2) , is a subsystem $(S, \Omega_2 \upharpoonright S)$, in which S is a Ω_2 -invariant subspace of H_2 and the restriction $\Omega_2 \upharpoonright S$ of Ω_2 to S has multiplicity 1. A string decomposition of (H_2, Ω_2) is an expression of (H_2, Ω_2) as a direct sum of strings:*

$$H_2 = \bigoplus_{j=1}^M H_{2j}, \quad \Omega_2 = \bigoplus_{j=1}^M \Omega_2 \upharpoonright H_{2j}, \tag{37}$$

in which each $(H_{2j}, \Omega_2 \upharpoonright H_{2j})$ is a string in (H_2, Ω_2) .

Evidently, the isomorphism (36) gives a string decomposition of H_2 . In view of the corresponding representation of Ω_2 as multiplication by the independent variable ω , construction of a decomposition of H_2 into strings is accomplished abstractly as follows: Choose a vector $v_1 \in H_2$ of maximal spectral type with respect to Ω_2 and obtain $H_{21} = \mathcal{O}_{\Omega_2}(v_1) \cong L^2(\mathbb{R}, \mathbb{C}, d\mu_1(\omega))$. Then, if $\mathcal{O}_{\Omega_2}(v_1) \neq H_2$, choose a vector v_2 of maximal spectral type in $\mathcal{O}_{\Omega_2}(v_1)^\perp$ and obtain $H_{22} = \mathcal{O}_{\Omega_2}(v_2) \cong L^2(\mathbb{R}, \mathbb{C}, d\mu_2(\omega))$, and so on. This infinite iterative process will produce a direct sum of the form (37). However, if the vectors v_n are chosen at will, this sum may not be all of H_2 : it may have an orthogonal complement, within H_2 , in which Ω_2 has uniform infinite multiplicity. One must be sure to include this part in the string decomposition. The structure provided by (36) shows that this is indeed possible.

The number M (which may be infinite) in the spectral representation (36) is the multiplicity of the operator Ω_2 ; M is the maximal multiplicity of any of the spectral values of Ω_2 .

Discussion. The coupling of the observable variables H_1 to the strings is accomplished through the coupling channels defined by Γ . Of course, there is in general no relation between a given decomposition of H_2 into strings and the coupling channels. If the strings can be chosen in such a way that the coupling channels split into two sets, one of which couples into one set of strings and the other of which couples into the complementary set of strings, and the H_1 -members of the two sets of channels are contained in orthogonal orbits of Ω_1 , then the open system $(H_1, \Omega_1, a(t))$ is decomposed into decoupled systems. We pursue a detailed study of the decoupling of open systems using their conservative extensions in Sec. 5.

In a typical example in which it is known that the open system $(H_1, m, A_1, a(t))$ is obtained naturally as the restriction of the dynamics of a given larger conservative system (\mathcal{H}, Ω) to a subspace of observable variables $H_1 \subset \mathcal{H}$ (as the open system of electromagnetic fields in a lossy medium or a crystalline solid in contact with a heat bath), the given conservative system is not necessarily minimal. The space of hidden variables for the minimal extension is actually a subspace of the given $H_2 = \mathcal{H} \ominus H_1$. We call this subspace the “coupled” part of H_2 and denote it by H_{2c} . H_{2c} coincides with H_2 if (\mathcal{H}, Ω) is minimal.

One of our main results concerns the situation in which there exist only finitely many simple coupling channels. This is the case that the rank of Γ is finite, such as in lattice systems, as we discuss in some detail as a motivating example in the following section. The result gives quantitative information about the size of H_{2c} within H_2 . As H_1 acts as the “hidden” variables for a hypothetical observer in H_2 , we have also an analogous result about the size of H_{1c} within H_1 , where H_{1c} is the part of H_1 that is reconstructible from the dynamics restricted to H_2 (or H_{2c}):

The (minimal) number of strings needed to extend an open system to a conservative one is no greater than the number of independent simple coupling channels between the spaces of observable and hidden variables.

The coupled part of the observable variable space has multiplicity (with respect to Ω_1) that is no greater than the number of independent simple coupling channels between the spaces of observable and hidden variables.

This result is stated precisely as Theorem 15 in Sec. 4, and its significance is discussed in Sec. 3.

3. OPEN SYSTEMS AND FROZEN DEGREES OF FREEDOM

We illustrate through a quite concrete example that certain degrees of freedom of a DD system can be “frozen”: they are not affected by the interaction with the hidden variables that causes the energy-dissipation effects. Thus a component of the state space that is “decoupled” from the hidden variables evolves conservatively, independent of the “coupled” DD part. This section may serve as a motivation for our detailed study in Sec. 4.

Consider a crystalline solid in contact with a heat bath. It has been observed that certain degrees of freedom of the solid do not contribute to its specific heat (Ref. 4, Sec. 3.1), (Ref. 5, Sec. 6.4). The calculation of the specific heat by the Dulong–Petit law is based on the law of equipartition of energy and the number of degrees of freedom. For that calculation to agree with the experiment, one has to leave out some degrees of freedom as if they were “frozen” and cannot be excited by the heat bath. In other words, there are system motions which are completely decoupled from the solid and heat bath interaction—they cannot be reached through the combination of surface contact and internal dynamics of the solid.

To find a sufficiently general scenario for such frozen degrees of freedom we consider an open system described by the variable $v_1 \in H_1$ as a part of the conservative system (6). We notice then that it is conceivable that the open system has a part not coupled to its exterior. In other words, there is an orthogonal decomposition

$$H_1 = H_{1c} \oplus H_{1d}, \quad (38)$$

where the subspaces H_{1c} and H_{1d} correspond to states coupled to and decoupled from the hidden variable $v_2 \in H_2$. To figure out the decomposition (38) we set $m_1 = 1$ in (6) (the general case is reduced to this one by proper renormalization of v_1), and consider the system

$$\begin{aligned} \partial_t v_1(t) &= -i\Omega_1 v_1(t) - i\Gamma_1 v_2(t), & \Omega_1 &= \Omega_1^\dagger \\ \partial_t v_2(t) &= -i\Gamma_1^\dagger v_1(t) - i\Omega_2 v_2(t) + f_2(t), & \Omega_2 &= \Omega_2^\dagger. \end{aligned} \quad (39)$$

The system (39) allows one to single out states v_1 which can be excited by the variables v_2 , which constitute subspace H_{1c} , namely

$$H_{1c} = \mathcal{O}_{\Omega_1}(\text{Ran } \Gamma) \text{ and, consequently, } H_{1d} = H_1 \ominus H_{1c}. \tag{40}$$

Based on this representation we deduce a condition that implies the existence of decoupled states H_{1d} in the presence of high symmetry in the internal dynamics in H_1 (corresponding to high multiplicity of Ω_1):

$$\text{mult}(\Omega_1 \upharpoonright H_{1c}) \leq \text{rank} \Gamma, \tag{41}$$

where $\text{mult} \{ \cdot \}$ and $\text{rank} \{ \cdot \}$ are the spectral multiplicity and the rank of an operator. We prove this inequality later on in Theorem 15. If Ω_1 and Γ are generic, the inequality (41) would also be necessary for the existence of decoupled states. We will refer to the condition (41) as the *spectral multiplicity condition*. This condition (41) readily implies that *an open system with low rank coupling and large spectral multiplicity must have decoupled (frozen) states*.

Below we construct a couple of simple examples of Hamiltonian open systems having decoupled degrees of freedom. A detailed discussion with theorems on the coupled and decoupled parts of the state variables is presented in Sec. 4.

3.1. An Oscillatory System with Frozen Degrees of Freedom

Let us consider an open oscillatory Hamiltonian system S_1 described by momentum and coordinate variables $\{p, q\}$ with $p, q \in \mathbb{R}^N$, where N is finite natural number. Hence, the Hilbert space of observable variables here is $H_1 = \mathbb{R}^{2N}$. We assume this open system to be a part of a larger Hamiltonian system for which the complimentary system S_2 of hidden degrees of freedom is described by variables $\{\pi, \varphi\}$ with $\pi, \varphi \in G$, where G is a real Hilbert space, and, hence, $H_2 = G \oplus G$. We don't write it explicitly, but rather presume that the system evolves according to the Hamilton equations with the total Hamiltonian to be of the form

$$H(p, q; \pi, \varphi) = h_1(p, q) + h_2(\pi, \varphi) + h_{\text{int}}(q, \varphi) \tag{42}$$

where h_1 and h_2 are correspondingly the internal energies of systems S_1 and S_2 , and h_{int} is the interaction energy between S_1 and S_2 . We assume h_1 and h_{int} to be of the form

$$h_1(p, q) = \frac{(p, p)}{2m} + \frac{\xi(q, q)}{2}, \quad h_{\text{int}}(q, \varphi) = \sum_{j=1}^J [(q, \gamma_{1j}) - (\varphi, \gamma_{2j})]^2 \tag{43}$$

where m and ξ are positive constants, $1 \leq J < N$, $\gamma_{1j} \in \mathbb{R}^N$ and $\gamma_{2j} \in G$. Evidently we can always choose an orthonormal system of vectors $\{\tilde{e}_1, \dots, \tilde{e}_N\}$ in

\mathbb{R}^N so that

$$E_\gamma = \text{span}\{\gamma_{11}, \dots, \gamma_{1J}\} = \text{span}\{\tilde{e}_{N-J+1}, \dots, \tilde{e}_N\}, \tag{44}$$

and introduce the corresponding new variables $\tilde{p}, \tilde{q} \in \mathbb{R}^N$ by

$$q = \sum_{s=1}^N q_s e_s = \sum_{s=1}^N \tilde{q}_s \tilde{e}_s \text{ where } \{e_1, \dots, e_N\} \text{ is the standard basis in } \mathbb{R}^N. \tag{45}$$

Next we introduce an orthogonal decomposition

$$\tilde{p} = \tilde{p}'' \oplus \tilde{p}', \tilde{q} = \tilde{q}'' \oplus \tilde{q}', \text{ where } \tilde{p}'', \tilde{q}'' \in E_\gamma \text{ and } \tilde{p}', \tilde{q}' \in \mathbb{R}^N \ominus E_\gamma, \tag{46}$$

and recast the energies in (43) as follows

$$\begin{aligned} h_1(p, q) &= h'_1(\tilde{p}', \tilde{q}') + h''_1(\tilde{p}'', \tilde{q}''), \text{ where} \tag{47} \\ h'_1(\tilde{p}', \tilde{q}') &= \frac{(\tilde{p}', \tilde{p}')}{2m} + \frac{\xi(\tilde{q}', \tilde{q}')}{2}, h''_1(\tilde{p}'', \tilde{q}'') = \frac{(\tilde{p}'', \tilde{p}'')}{2m} + \frac{\xi(\tilde{q}'', \tilde{q}'')}{2}, \\ h_{\text{int}}(q, \varphi) &= \sum_{j=1}^J [(\tilde{q}'', \tilde{\gamma}_{1j}) - (\varphi, \gamma_{2j})]^2. \end{aligned}$$

It is evident from (47) that the variables $\{\tilde{p}', \tilde{q}'\}$ are decoupled from the system S_2 , and all the coupling from S_1 to S_2 is only through the variables $\{\tilde{q}''\}$. In fact, S_1 couples directly to S_2 through the variables \tilde{q}'' only, but this coupling affects \tilde{p}'' through the internal dynamics in S_1 . $\{\tilde{p}', \tilde{q}'\}$ remain, however, unaffected. This together with (46) yields the following estimates for the space H_{1d} of “decoupled” states $\{\tilde{p}', \tilde{q}'\}$.

$$H_{1d} \supseteq (\mathbb{R}^N \ominus E_\gamma)^2, \text{ and, hence, } \dim H_{1d} \geq 2(N - J). \tag{48}$$

An elementary analysis of the used arguments shows that the existence of decoupled variables in the above example is due to (i) the highly symmetric form of the Hamiltonian $h_1(p, q)$ in (43), resulting in the maximal spectral multiplicity N , and (ii) the coupling of rank J , which is less than N and application of the spectral multiplicity condition (41).

Notice that, if instead of (43), we would have

$$h_1(p, q) = \sum_{s=1}^N \frac{p_s^2}{2m_s} + \sum_{s=1}^N \frac{\xi_s q_s^2}{2} \tag{49}$$

with all different and generic m_s and ξ_s , then the corresponding spectral multiplicity would be one and there will be no decoupled degrees of freedom. We point out also that, in this case, for a generic γ_{1j} in the representation (43) every vector from the original orthonormal system e_1, \dots, e_N in \mathbb{R}^N has nonzero projections onto both E_γ and $\mathbb{R}^N \ominus E_\gamma$, implying that *generically none of the original variables*

$\{p_s, q_s\}$ can be considered as being decoupled from the system S_2 . This indicates that decoupling of variables due the spectral multiplicity, though elementary, is not trivial.

3.2. Toy Model of a Solid with Frozen Degrees of Freedom

We construct here a toy model for a solid having frozen degrees of freedom due to high spectral multiplicity, naturally arising from system symmetries. Let us consider the d -dimensional lattice

$$\mathbb{Z}^d = \{n : n = (n_1, \dots, n_d), n_j \in \mathbb{Z}\} \text{ where } \mathbb{Z} \text{ is the set of integers,} \quad (50)$$

and introduce a system S as a lattice array of identical oscillatory systems similar to that described in the previous section. Namely, we assume that the system state is of the form $u = \{[p_n, q_n], n \in \mathbb{Z}^d\}$ where with $p_n, q_n \in \mathbb{R}^N$, where N is a finite natural number.

The system Hamiltonian $H(p, q)$ is assumed to be spatially homogeneous, local, and of the form

$$H(p, q) = \sum_{n \in \mathbb{Z}^d} \left[h_1(p_n, q_n) + \sum_{j=1}^J \|(\nabla q_n, \gamma_j)\|^2 \right], p, q \in H, \quad (51)$$

where the local Hamiltonian $h_1(p, q)$ is defined by (43), and the vectors $\gamma_j \in \mathbb{R}^N, 1 \leq j \leq J$, describe the interactions between neighboring sites through the discrete gradient ∇ . An expansion of the inner sum gives

$$\sum_{j=1}^J \|\nabla_n(q_n, \gamma_j)\|^2 = \sum_{j=1}^J \sum_{i=1}^d ((q_n, \gamma_j) - (q_{n+e_i}, \gamma_j))^2, \quad (52)$$

in which $e_i = (\delta_{i1}, \dots, \delta_{id})$. Now denoting

$$|m|_0 = \max_{1 \leq j \leq d} |m_j|, m = (m_1, \dots, m_d) \in \mathbb{Z}^d \quad (53)$$

we consider an arbitrary finite lattice cube

$$\Lambda = \Lambda_L = \{n \in \mathbb{Z}^d : |n|_0 \leq L\} \text{ where } L \geq 2 \text{ is an integer,} \quad (54)$$

and define its volume $|\Lambda|$ by

$$|\Lambda| = \text{number of sites } n \in \Lambda. \quad (55)$$

Now we introduce a system S_Λ associated with the finite lattice cube Λ , in which the states are functions from Λ to $\mathbb{R}^N \oplus \mathbb{R}^N$, or $\{[p_n, q_n], n \in \Lambda\}$, with the

Hamiltonian

$$H_\Lambda(p, q) = \sum_{n \in \Lambda} \left[h_1(p_n, q_n) + \sum_{j=1}^{J_0} \|\nabla q_n, \gamma_j\|^2 \right]; \quad q_n = 0 \text{ for } n \notin \Lambda. \quad (56)$$

Recall now that the system dynamics is described then by the Hamilton equations

$$\frac{dp}{dt} = -\frac{dH}{dq}, \quad \frac{dq}{dt} = \frac{dH}{dp}, \quad (57)$$

which, in our case, turns into the linear evolution equation of the form

$$\frac{du}{dt} = -i\Omega_u, \quad u = [p, q], \quad (58)$$

in which multiplication by i is defined by $i[p, q] = [-q, p]$. Without writing the relevant operator (matrix) Ω explicitly, we simply denote by Ω_Λ the respective matrix for the Hamiltonian H_Λ . Observe now that in view of the form (56) of the Hamiltonian H_Λ , an open oscillatory system associated with any single site $n \in \Lambda$ is exactly of the form considered in the previous section (see (51) and (52)), and, consequently, it has decoupled degrees of freedom described by the space $(\mathbb{R}^N \ominus E_\gamma)^2$ not depending on n . This implies that the space of decoupled (frozen) states H_d satisfies

$$H_d \supseteq (\mathbb{R}^N \ominus E_\gamma)^{2|\Lambda|}. \quad (59)$$

If we put $H = H_{1d} \oplus H_{1c}$, then this, combined with the fact the local Hamiltonians (h_1) at all sites are identical gives the rank of the coupling as $J|\Lambda|$. By the spectral multiplicity condition (41) we then obtain a bound on the spectral multiplicity of the restriction $\Omega_\Lambda \upharpoonright H_c$ to the “coupled” part H_c :

$$\text{mult}(\Omega_\Lambda \upharpoonright H_c) \leq J|\Lambda| = J(2L + 1)^d. \quad (60)$$

In fact, a more elaborate analysis based on introduction of lattice toruses along with lattice cubes can produce an approximate formula of the following form

$$\frac{\text{mult}\Omega_\Lambda}{|\Lambda|} = C_0 + O(|\Lambda|^{-\frac{1}{d}}), \quad (61)$$

where C_0 is a constant similar to J .

The estimates (60) and (61) for the solid toy model indicate that the high spectral multiplicity can cause many degrees of freedom to be completely decoupled from the rest of the system.

4. RECONSTRUCTIBILITY FROM OPEN SUBSYSTEMS

According to our general strategy, we regard an open system within its conservative system from two points of view that are closely related. In the first,

we consider a conservative system composed of two coupled subsystems, each treated equally. Because of the coupling, the subsystems are open, and we analyze the extent to which the conservative system is reconstructible from either of its open subsystems. In the second, the objects that play the leading role are the “master” conservative system and a given subsystem. The subsystem is open, as it interchanges energy with the master system.

4.1. Two Coupled Open Systems

We first investigate a conservative system composed of two coupled open ones. Typically, one system will be observable, such as a resonator, and the other will represent its “exterior” which we associate with the hidden degrees of freedom.

Let us begin with two conservative systems, Closed System 1, identified by the triple (H_1, m_1, A_1) that represents an observable system

$$m_1 \partial_t v_1(t) = -iA_1 v_1(t) + f_1(t) \quad \text{in } H_1 \quad (62)$$

satisfying the rest condition (Condition 2, p. 8), and another linear system, Closed System 2, identified by the triple (H_2, m_2, Ω_2) that represents the system of hidden variables

$$m_2 \partial_t v_2(t) = -iA_2 v_2(t) + f_2(t) \quad \text{in } H_2 \quad (63)$$

also satisfying the rest condition. A_1 and A_2 are self-adjoint, and the mass operators m_1 and m_2 are positive. We then couple the two systems through a bounded operator Γ and its adjoint:

$$\Gamma : H_2 \rightarrow H_1, \quad \Gamma^\dagger : H_1 \rightarrow H_2. \quad (64)$$

The conservative system as composed of these two subsystems then has the form

$$\begin{aligned} m_1 \partial_t v_1(t) &= -iA_1 v_1(t) - i\Gamma v_2(t) + f_1(t), \\ m_2 \partial_t v_2(t) &= -i\Gamma^\dagger v_1(t) - iA_2 v_2(t) + f_2(t). \end{aligned} \quad (65)$$

Using the following rescaling transformation

$$v_j \rightarrow m_j^{-\frac{1}{2}} v_j, \quad A_j \rightarrow m_j^{\frac{1}{2}} \Omega_j m_j^{\frac{1}{2}}, \quad f_j \rightarrow m_j^{\frac{1}{2}} f_j, \quad \Gamma \rightarrow m_1^{\frac{1}{2}} \Gamma m_2^{\frac{1}{2}} \quad (66)$$

we recast the system (65) into the simpler form

$$\begin{aligned} \partial_t v_1(t) &= -i\Omega_1 v_1(t) - i\Gamma v_2(t) + f_1(t), \\ \partial_t v_2(t) &= -i\Gamma^\dagger v_1(t) - i\Omega_2 v_2(t) + f_2(t), \end{aligned} \quad (67)$$

which we will use from now on. We refer to the operators Ω_1 and Ω_2 as the *frequency operators* for the observable and hidden systems.

In matrix form, the system (67) is written as

$$\partial_t \mathcal{V} = -i\Omega \mathcal{V} + \mathcal{F}, \quad \mathcal{V}, \mathcal{F} \in \mathcal{H} = H_1 \oplus H_2, \tag{68}$$

in which the frequency operator Ω has the block-matrix structure

$$\Omega = \begin{bmatrix} \Omega_1 & \Gamma \\ \Gamma^\dagger & \Omega_2 \end{bmatrix}. \tag{69}$$

Because of the coupling, both systems become open. Their dynamics are obtained by projecting the dynamics of the large conservative system in \mathcal{H} to H_1 and H_2 separately. For System 1, this means setting the forcing from the second equation of the system (67) to zero ($f_2(t) = 0$), solving for v_2 , and then inserting the result into the first equation. This, together with an analogous computation for System 2, results in the dynamical equations for the open systems Open System 1 ($H_1, \Omega_1, a_1(t)$) and Open System 2 ($H_2, \Omega_2, a_2(t)$):

$$\partial_t v_1(t) = -i\Omega_1 v_1(t) - \int_0^\infty a_1(\tau) v_1(t - \tau) d\tau + f_1(t) \quad \text{in } H_1, \tag{70}$$

$$\partial_t v_2(t) = -i\Omega_2 v_2(t) - \int_0^\infty a_2(\tau) v_2(t - \tau) d\tau + f_2(t) \quad \text{in } H_2, \tag{71}$$

in which

$$a_1(t) = \Gamma e^{-i\Omega_2 t} \Gamma^\dagger \quad \text{and} \quad a_2(t) = \Gamma^\dagger e^{-i\Omega_1 t} \Gamma, \tag{72}$$

and the functions $a_j(t)$ are the friction functions. The rest condition, Condition 2 continues to hold, and, by virtue of their form, the equations automatically satisfy the power dissipation condition, Condition 1.

We ask the question: How much of H_2 can be reconstructed from Open System 1 (70) alone; in other words, how much information about the hidden variables is encoded in the friction function $a_1(t)$ for the observable variables? We can view $a_1(t)$ as a dynamical mechanism by which an observer confined to the observable state variables detects or influences the hidden degrees of freedom. The subspace of H_2 that is reconstructible by $a_1(t)$ we call the *coupled component of H_2* and denote it by H_{2c} . Clearly this subspace is determined by the coupling channels to H_2 given by Γ^\dagger and the internal action by Ω_2 on H_2 ; this is explicitly evident in the form $a_1(t) = \Gamma e^{-i\Omega_2 t} \Gamma^\dagger$.

The question of the extent to which the observable system determines the hidden is tantamount to that of *determining the unique minimal conservative extension of Open System 1 within the large system (\mathcal{H}, Ω)* (see Sec. 2). According to Proposition 3 the Hilbert state space $\mathcal{H}_{\min} \supset H_1$ of this conservative extension is simply the orbit (see Definition 5) of H_1 under the action of Ω , as a subspace of \mathcal{H} :

$$\mathcal{H}_{\min} = \mathcal{O}_\Omega(H_1) = H_1 \oplus H_{2c}, \quad H_{2c} := \mathcal{H}_{\min} \ominus H_1. \tag{73}$$

In addition, it can be shown that $H_{2c} \subseteq H_2$ is invariant under the action of Ω_2 , namely

$$\Omega_2 H_{2c} \subseteq H_{2c} \subseteq H_2, \tag{74}$$

and we refer to H_{2c} as the *coupled component of H_2* . This construction of $H_{2c} \subseteq H_2$ gives rise to a canonical Ω_2 -invariant orthogonal decomposition of the hidden state variables,

$$H_2 = H_{2c} \oplus H_{2d}. \tag{75}$$

We refer to the subspace H_{2d} defined in (75) as the *decoupled component of H_2* . The systems in H_{2c} and H_{2d} evolve independently in time by the internal dynamics Ω_2 of the hidden variables H_2 . Furthermore, since it can be shown that the operator range $\text{Ran}(\Gamma^\dagger) \subset H_{2c}$, we see that no forcing function from H_{2d} can influence the dynamics of the observable variables and, in turn, does not influence the dynamics of H_{2c} through the coupling.

In an analogous way, we can ask, how much of H_1 can a hypothetical observer confined to H_2 reconstruct? The component H_{1c} of H_1 that is reconstructible by an observer in the hidden state variables we call the *coupled component of H_1* . Its orthogonal complement H_{1d} is the *decoupled component of H_1* , and we have the decomposition

$$H_1 = H_{1c} \oplus H_{1d} \tag{76}$$

which is invariant under the action of Ω_1 . The state space of the unique minimal conservative extension of Open System 2 is $H_{1c} \oplus H_2$.

With respect to these decompositions of the observable and hidden variables into the coupled and decoupled components $\mathcal{H} = H_{1d} \oplus H_{1c} \oplus H_{2c} \oplus H_{2d}$, the frequency operator Ω for the conservative system (67) has the matrix form

$$\Omega = \begin{bmatrix} \Omega_{1d} & 0 & 0 & 0 \\ 0 & \Omega_{1c} & \Gamma_c & 0 \\ 0 & \Gamma_c^\dagger & \Omega_{2c} & 0 \\ 0 & 0 & 0 & \Omega_{2d} \end{bmatrix}, \tag{77}$$

in which the subscripts refer to retrictions of the domain:

$$\Omega_{ic} = \Omega \upharpoonright H_{ic}, \quad \Omega_{id} = \Omega \upharpoonright H_{id}, \quad i = 1, 2, \quad \text{and} \quad \Gamma_c = \Gamma \upharpoonright H_{2c}. \tag{78}$$

We can see from (77) that the decoupled parts H_{1d} and H_{2d} , can be analyzed independently of the rest of the system justifying their name “decoupled.” Furthermore, the conservative subsystem $(H_{1c} \oplus H_{2c}, \Omega_c)$ with frequency operator

$$\Omega_c = \begin{bmatrix} \Omega_{1c} & \Gamma_c \\ \Gamma_c^\dagger & \Omega_{2c} \end{bmatrix}, \tag{79}$$

which consists of the part of H_1 reconstructible by Open System 2 alone and the part H_2 reconstructible by Open System 1 alone, is itself fully reconstructible by either of the open subsystems $(H_{1c}, \Omega_{1c}, a_1(t))$ or $(H_{2c}, \Omega_{2c}, a_2(t))$. This is equivalent to the statement that $(H_{1c} \oplus H_{2c}, \Omega_c)$ is the unique minimal conservative extension, realized as a subsystem of (\mathcal{H}, Ω) , of each of its open components separately. This motivates the following definition.

Definition 8. (reconstructibility) *A conservative linear system composed of two coupled subsystems*

$$\partial_t v_1(t) = -i\Omega_1 v_1(t) - i\Gamma v_2(t) + f_1(t),$$

$$\partial_t v_2(t) = -i\Gamma^\dagger v_1(t) - i\Omega_2 v_2(t) + f_2(t),$$

with $v_1(t) \in H_1$ and $v_2(t) \in H_2$ is called reconstructible if it is the minimal conservative extension of each of the open projected linear systems

$$\partial_t v_1(t) = -i\Omega_1 v_1(t) - \int_0^\infty \Gamma e^{-i\Omega_2 \tau} \Gamma^\dagger v_1(t - \tau) d\tau + f_1(t) \quad \text{in } H_1,$$

$$\partial_t v_2(t) = -i\Omega_2 v_2(t) - \int_0^\infty \Gamma^\dagger e^{-i\Omega_1 \tau} \Gamma v_2(t - \tau) d\tau + f_2(t) \quad \text{in } H_2.$$

In other words, the conservative system $(H_1 \oplus H_2, \Omega)$ (Ω is defined by its decomposition (69)) is reconstructible if all of H_2 can be reconstructed from the open system $(H_1, \Omega_1, a_1(t))$ and all of H_1 can be reconstructed from the open system $(H_2, \Omega_2, a_2(t))$.

We say that H_2 is reconstructible from the open system $(H_1, \Omega_1, a_1(t))$ if $H_1 \oplus H_2$ is (isomorphic to) the state space for the minimal conservative extension of $(H_1, \Omega_1, a_1(t))$.

A conservative system that is reconstructible may possibly be further decomposed into independent conservative subsystems that commute with the projection to H_1 , in other words, that are of the form $H'_1 \oplus H'_2$ with $H'_1 \subseteq H_1$ and $H'_2 \subseteq H_2$. However, H'_1 will not contain the entire range of Γ and accordingly will not contain all of the information of the delayed response function $a_1(t)$. We take up these finer decompositions further in Sec. 5.

The simplest reconstructibility theorem is as follows. More general reconstructibility statements as well as the proofs are given in Sec. 6.

Theorem 9. (system reconstructibility) *Let a conservative “master” system composed of two coupled systems be given:*

$$\partial_t v_1(t) = -i\Omega_1 v_1(t) - i\Gamma v_2(t) + f_1(t), \tag{80}$$

$$\partial_t v_2(t) = -i\Gamma^\dagger v_1(t) - i\Omega_2 v_2(t) + f_2(t), \tag{81}$$

with $v_1(t)$ in the state space H_1 and $v_2(t)$ in the state space H_2 , and let the coupling operator $\Gamma : H_2 \rightarrow H_1$ be bounded. Let also H_{2c} denote the subspace of H_2 that is reconstructible from Open System 1 and H_{1c} the subspace of H_1 that is reconstructible from Open System 2.

- i. H_{1c} consists of the set of states of H_1 that are accessible by applying the internal dynamics of H_1 (given by Ω_1) to all vectors in H_1 to which H_2 is directly coupled by Γ , that is,

$$H_{1c} = \mathcal{O}_{\Omega_1}(\text{Ran } \Gamma). \tag{82}$$

Similarly,

$$H_{2c} = \mathcal{O}_{\Omega_2}(\text{Ran } \Gamma^\dagger). \tag{83}$$

- ii. The restriction of the master system to $H_1 \oplus H_{2c}$ is the unique minimal conservative extension of H_1 , and the restriction to $H_{1c} \oplus H_2$ is the unique minimal conservative extension of H_2 .
- iii. The restriction of the master system to $H_{1c} \oplus H_{2c}$ is the unique reconstructible subsystem of the master system that completely determines the friction functions of Open Systems 1 and 2, namely $\Gamma e^{-i\Omega_2 t} \Gamma^\dagger$ and $\Gamma^\dagger e^{-i\Omega_1 t} \Gamma$.
- iv. $(H_1 \oplus H_2, \Omega)$ is reconstructible if and only if H_1 and H_2 have no nontrivial Ω -invariant subspaces.

Example: Two Coupled Finite Systems. With a simple finite-dimensional example of two coupled open systems, we illustrate the interaction between the two components and the extent to which each is determined, or reconstructible, by the other. The observations are generalized and proved in Theorem 9.

Let us begin with the state space of observable variables $H_1 = \mathbb{C}^2$, with variable vector $v \in H_1$ and an open DD system

$$\begin{aligned} \partial_t v(t) = & -i \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} v(t) - i \begin{bmatrix} \alpha \\ \beta \end{bmatrix} [\alpha^* \ \beta^*] \int_0^\infty (|\gamma|^2 e^{-i\mu_1 \tau} \\ & + |\delta|^2 e^{-i\mu_2 \tau}) v(t - \tau) d\tau, \end{aligned} \tag{84}$$

in which we assume $|\alpha|^2 + |\beta|^2 = 1$ and μ_1 and μ_2 are real. The operator for the internal dynamics in H_1 is

$$\Omega_1 = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}, \tag{85}$$

in which a, c, λ_1 , and λ_2 are real. The delayed-response function

$$ia_1(t) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} [\alpha^* \ \beta^*] (|\gamma|^2 e^{-i\mu_1 t} + |\delta|^2 e^{-i\mu_2 t}) \tag{86}$$

involves two frequencies, each with a matrix factor of rank one. The space H_2 of hidden variables is therefore isomorphic to \mathbb{C}^2 ; in fact,

$$ia_1(t) = \Gamma e^{-i\Omega_2 t} \Gamma^\dagger, \tag{87}$$

where

$$\Gamma = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} [\gamma^* \ \delta^*], \quad \Omega_2 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}. \tag{88}$$

The minimal conservative extension of $(H_1, \Omega_1, a_1(t))$ is (\mathcal{H}, Ω) , where

$$\mathcal{H} = H_1 \oplus H_2, \quad \Omega = \left[\begin{array}{cc|cc} a & b & \alpha\gamma^* & \alpha\delta^* \\ b^* & c & \beta\gamma^* & \beta\delta^* \\ \hline \alpha^*\gamma & \beta^*\gamma & \mu_1 & 0 \\ \alpha^*\delta & \beta^*\delta & 0 & \mu_2 \end{array} \right]. \tag{89}$$

In this particular example, Γ has rank 1 because the matrices in $ia_1(t)$ for the two frequencies have the same range. This range, which is the range of Γ , happens to be an eigenspace for Ω_1 corresponding to the eigenvalue λ_1 (Eq. 85). The delayed-response function for the reduced dynamics in H_2 therefore only involves this single frequency:

$$a_2(t) = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} [\gamma^* \ \delta^*] e^{i\lambda_1 t}. \tag{90}$$

The state space of the minimal conservative extension of $(H_2, \Omega_2, a_2(t))$ is the three-dimensional space

$$H_{1c} \oplus H_2, \tag{91}$$

in which H_{1c} is the ‘‘coupled component’’ of H_1 , consisting of the eigenspace for the eigenvalue λ .

If one projects the dynamics to H_{1c} , then its minimal conservative extension is the same as that of $(H_2, \Omega_2, a_2(t))$. Thus, the space H_2 , as well as its internal dynamics operator Ω_2 and the coupling Γ , are reconstructible from the dynamics projected to H_{1c} , just as H_{1c}, Ω_1 restricted to H_{1c} , and Γ are reconstructible from $(H_2, \Omega_2, a_2(t))$. We therefore call the system in $H_{1c} \oplus H_2$ *reconstructible* (Definition 8).

In Theorem 9, we prove that a coupled pair of open systems forming a conservative system, $H_1 \oplus H_2$ admits a unique reconstructible subsystem system $H_{1c} \oplus H_{2c}$, containing all the information of $a_1(t)$, in which the projection of the dynamics to each part is sufficient to reconstruct the other. As in the simple example of this subsection, it is always true that H_{1c} is the Ω_1 -orbit of $\text{Ran } \Gamma$ and H_{2c} is the Ω_2 -orbit of $\text{Ran } \Gamma^\dagger$.

4.2. Open Subsystems of Conservative Systems

Often an open system arises as a part of a given conservative system (\mathcal{H}, Ω) projected onto an observable subspace $H_1 \subset \mathcal{H}$. We investigate the way in which the state space \mathcal{H}_{\min} of the minimal conservative extension of the open system in H_1 is reconstructed within the spectral structure of (\mathcal{H}, Ω) . We shall see that \mathcal{H}_{\min} is generated by the projections of all vectors in H_1 onto the eigenspaces of Ω , as well as by the projections, onto the eigenspaces of Ω , of those vectors in H_1 and $\mathcal{H} \ominus H_1$ that are directly coupled through Γ (the ranges of Γ and Γ^\dagger). We discuss both points of view. We investigate similar constructions for the generation of H_{1c} and H_{2c} by eigenmodes of Ω_1 and Ω_2 and arrive at one of our main results, Theorem 15, which bounds the number of extending strings by the rank of the coupling.

In the case, say, of a finite resonator embedded within an infinite planar lattice, for which the multiplicity of each eigenvalue is infinite, this result has immediate consequences: namely, the multiplicities of the eigenfrequencies for the minimal extension $(\mathcal{H}_{\min}, \Omega|_{\mathcal{H}_{\min}})$ are uniformly bounded and hence \mathcal{H}_{\min} is but a very small part of \mathcal{H} . The perhaps more interesting point of view, in which the directly coupled modes (the ranges of Γ and Γ^\dagger) generate \mathcal{H}_{\min} , has special significance for an object in surface contact with an infinite medium.

4.2.1. Generating the Minimal Extension from the Observable States.

Not surprisingly, all of the modes (eigenfunctions) of the frequency operator Ω that contribute to the Ω -mode decomposition of any one of the vectors in the “observable” space H_1 must be included as states of the minimal extension. These modes, in turn, generate all of the observable vectors, and therefore the entire space \mathcal{H}_{\min} .

To understand this, let us begin with the case in which \mathcal{H} is finite dimensional. Let $\{\lambda_\alpha\}_{\alpha=1}^n$ be the distinct eigenvalues of Ω . We then have a decomposition of \mathcal{H} into orthogonal eigenspaces

$$\mathcal{H} = \bigoplus_{\alpha=1}^n \mathcal{H}^\alpha, \tag{92}$$

with respect to which the operator Ω is diagonal:

$$\text{for each } v = \sum_{\alpha=1}^n v_\alpha, \quad \Omega v = \sum_{\alpha=1}^n \lambda_\alpha v_\alpha. \tag{93}$$

As we have discussed, \mathcal{H}_{\min} is the subspace of \mathcal{H} that is generated by the vectors in H_1 through the operator Ω , in other words, it is the orbit $\mathcal{O}_\Omega(H_1)$ of H_1 under Ω . In the finite-dimensional case, this is simply the vector space spanned by

the vectors $\Omega^k(v)$ for $v \in H_1$ and $0 \leq k < n$. Since the monomials λ^k , restricted to the spectrum of Ω , span the space of functions defined on the spectrum, we have $v_\alpha \in \mathcal{H}_{\min}$ for each $\alpha = 1, \dots, n$. Now since the vector $v \in H_1$ is in turn generated through linear combination by its projections v_α , we conclude that \mathcal{H}_{\min} is generated by the projections of H_1 onto the eigenspaces of Ω , denoted by $\pi_\alpha(H_1)$:

$$\mathcal{H}_{\min} = \mathcal{O}_\Omega(H_1) = \bigoplus_{\alpha=1}^n \pi_\alpha(H_1). \tag{94}$$

Thus we have a spectral decomposition for $(\mathcal{H}_{\min}, \Omega \upharpoonright \mathcal{H}_{\min})$ explicitly in terms of subspaces of the eigenspaces for (\mathcal{H}, Ω) .

This result can be extended to the case in which \mathcal{H} is infinite-dimensional and Ω has pure point spectrum, say $\{\lambda_\alpha\}_{\alpha=1}^\infty$ (Theorem 13). In this case, the class of polynomial functions p of the operator Ω of degree less than n , which was sufficient for the finite-dimensional case, must be expanded to include all continuous functions $f(\Omega)$ as understood in the classical functional calculus. Applying all continuous functions of Ω to all vectors in H_1 and taking the closure gives the orbit $\mathcal{O}_\Omega(H_1)$, and the projections $\pi_\alpha(H_1)$ onto the eigenspaces are contained in this orbit. We obtain again the spectral decomposition (94), in which $n = \infty$.

A typical frequency operator Ω for a conservative dynamical system does not have pure point spectrum, and we now face the problem of extending this construction of \mathcal{H}_{\min} to the case of general spectrum. The fairly simple construction of \mathcal{H}_{\min} we have discussed for pure point spectrum provides us with the correct principle:

Rule 10. (*extension by modes of Ω via observable variables*) The minimal conservative extension of $(H_1, \Omega \upharpoonright H_1, a_1(t))$ is generated by linear superposition, within the given master system (\mathcal{H}, Ω) , by all of the modes that appear in the eigenmode decomposition, into eigenmodes of Ω , of any of the states of H_1 .

When Ω has continuous spectrum, its modes are no longer finite-norm (finite-energy) states—they no longer exist as elements of the Hilbert space \mathcal{H} . In this case, it one can replace the projections $\pi_\alpha(H_1)$ of H_1 to the eigenspaces of Ω with a set of spectral projections associated with Ω :

$$\left\{ \pi(H_1) : \pi = \int_\Delta dE_\lambda \text{ for some interval } \Delta \text{ of } \mathbb{R} \right\}, \tag{95}$$

in which dE_λ is the spectral resolution of the identity associated with Ω . This set generates \mathcal{H}_{\min} by linear combination and closure. Of course, the spaces $\pi(H_1)$ are in general no longer orthogonal to each other for two different choices of the

projection π , so we no longer have an orthogonal decomposition as in Eq. (94). The projections can be localized to include only spectral intervals of length ϵ for arbitrarily small ϵ , so that one approaches spectrally localized projections, nearly representing eigenmode spaces, as $\epsilon \rightarrow 0$. However, the projections no longer make sense for $\epsilon = 0$ (unless the spectrum has no continuous part).

Fortunately, one need not abandon the use of modes altogether when dealing with continuous spectra. Just as the Laplace operator $-\Sigma_i \partial_{x_i x_i}$ has the extended states $e^{i\lambda x}$ for its modes, which generate all sufficiently regular functions through integral superposition, a proper treatment of modes of Ω and decomposition of states into these modes is accomplished by a furnishing of $\mathcal{H} : \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$. The modes lie in the larger Hilbert space \mathcal{H}_- , endowed with a smaller norm, with respect to which \mathcal{H} is dense in \mathcal{H}_- . All elements of the smaller space \mathcal{H}_+ , which is dense in \mathcal{H} , are represented as integral superpositions of the modes:

$$v = \int \Psi_v(\lambda) d\mu, \tag{96}$$

in which $d\mu$ is a spectral measure for Ω , Ψ is a $d\mu$ -measureable function with values in \mathcal{H}_- , and $\Psi(\lambda)$ is a mode for Ω for the frequency λ . This means that $\langle \Psi(\lambda) | \Omega | v \rangle = \lambda \langle \Psi(\lambda) | v \rangle$ whenever all of these objects are defined. See, for example, Ref. 8.

With this structure, each state $v \in H_1 \cap \mathcal{H}_+$ is decomposed into its modes $\Psi_v(\lambda)$, where λ runs over all spectral values. Integral superpositions of these modes then generate \mathcal{H}_{\min} :

$$\mathcal{H}_{\min} = \left\{ \int \Psi(\lambda) d\mu : \Psi \in L^2(\mathbb{R}, \mathcal{H}_-, d\mu), \forall \lambda \in \mathbb{R} \exists v \in H_1, \Psi(\lambda) = \Psi_v(\lambda) \right\}. \tag{97}$$

A rigorous treatment of generalized modes is quite technical; we do not pursue it in this work but leave it for a forthcoming work in which we treat unbounded coupling.

4.2.2. Generating the Conservative Extension from the Coupling Channels

The role of the coupling operator Γ and the space of hidden variables $\mathcal{H} \ominus H_1$ is not emphasized in the construction of \mathcal{H}_{\min} from H_1 by the action of Ω . We shall now show an alternative way to generate \mathcal{H}_{\min} , provided $H_1 = H_{1c}$, that is, provided that H_1 has no Ω -invariant subspaces. This is by the action of Ω on $\text{Ran } \Gamma$ and $\text{Ran } \Gamma^\dagger$, or by the action of Ω on the coupling channels (see Definition 6).

We shall see that the minimal reconstructible system described in Theorem 9 is obtained by linear superposition of the projections of the ranges of Γ and Γ^\dagger onto the eigenspaces of Ω . This can also be expressed as linear superposition of

those modes of Ω that are present in the mode decompositions of all vectors in $\text{Ran } \Gamma$ and $\text{Ran } \Gamma^\dagger$.

To understand why this is true, let us decompose Ω into a diagonal part representing the internal dynamics of the observable and hidden variables and a part representing the coupling:

$$\Omega = \overset{\circ}{\Omega} + \overset{\circ}{\Gamma}, \quad \overset{\circ}{\Omega} = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix}, \quad \overset{\circ}{\Gamma} = \begin{bmatrix} 0 & \Gamma \\ \Gamma^\dagger & 0 \end{bmatrix}. \quad (98)$$

Since $\text{Ran } \overset{\circ}{\Gamma}$ is contained in $\mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma})$ and $\Omega = \overset{\circ}{\Omega} + \overset{\circ}{\Gamma}$, the operator Ω generating the orbit can be replaced by $\overset{\circ}{\Omega}$: Recalling the definition of the orbit of a subset of a Hilbert space under the action of two operators (22), we obtain

$$\mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\overset{\circ}{\Omega}, \overset{\circ}{\Gamma}}(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\overset{\circ}{\Omega}, \overset{\circ}{\Gamma}}(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\overset{\circ}{\Omega}}(\text{Ran } \overset{\circ}{\Gamma}). \quad (99)$$

Now, by the decoupling of the action of $\overset{\circ}{\Omega}$ with respect to the decomposition $H_1 \oplus H_2$ and the splitting $\text{Ran } \overset{\circ}{\Gamma} = \text{Ran } \Gamma \oplus \text{Ran } \Gamma^\dagger$, we obtain a simple characterization of this orbit:

$$\mathcal{O}_{\overset{\circ}{\Omega}}(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\overset{\circ}{\Omega}}(\text{Ran } \Gamma) \oplus \mathcal{O}_{\overset{\circ}{\Omega}}(\text{Ran } \Gamma^\dagger) = \mathcal{O}_{\Omega_1}(\text{Ran } \Gamma) \oplus \mathcal{O}_{\Omega_2}(\text{Ran } \Gamma^\dagger). \quad (100)$$

Assuming that $H_1 = H_{1c}$, or, equivalently, that H_1 has no nontrivial Ω -invariant subspace (see Theorem 9, part (i)), we obtain

$$\mathcal{H}_{\min} = \mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma}). \quad (101)$$

We may now adapt our previous discussion concerning the construction of \mathcal{H}_{\min} from H_1 in order to understand the construction of \mathcal{H}_{\min} from $\text{Ran } \overset{\circ}{\Gamma}$ within the spectral structure of (\mathcal{H}, Ω) simply by replacing H_1 in the arguments with $\text{Ran } \overset{\circ}{\Gamma}$.

Rule 11. (extension by modes of Ω via coupling channels) If H_1 contains no Ω -invariant subspace, then the minimal conservative extension of $(H_1, \Omega \upharpoonright H_1, a_1(t))$ is generated by linear superposition, within the given master system (\mathcal{H}, Ω) , by all of the modes that appear in the eigenmode decomposition, into eigenmodes of Ω , of any of the states in the ranges of Γ and Γ^\dagger .

The representations (94) and (97) with $\text{Ran } \overset{\circ}{\Gamma}$ replacing H_1 are valid for the pure point and general cases, respectively. In the case of pure point spectrum, in particular for finite systems, we can reformulate Rule 11 in terms of a characterization of reconstructibility:

$(\mathcal{H} = H_1 \oplus H_2, \Omega)$ is reconstructible if and only if, on each mode of Ω , Γ does not vanish.

4.2.3. Generating H_1 and H_2 from Γ

These rules can be applied as well to the two components of the reconstructible system $(\mathcal{H}' = H_{1c} \oplus H_{2c}, \Omega \upharpoonright \mathcal{H}')$ from the operators of their internal dynamics, Ω_1 and Ω_2 . Recall from part (i) of Theorem 9 that the “coupled” part H_{1c} of H_1 is generated through the action of the frequency operator Ω_1 on the range of Γ . We obtain therefore, by the same reasoning as before, results on the construction of H_{1c} by superposition of the modes of Ω_1 obtained from the projections of $\text{Ran } \Gamma$ onto the eigenspaces of Ω_1 . Of course, this applies equally to the construction of H_{2c} by modes of Ω_2 .

Rule 12. (Generating H_{1c} and H_{2c}) The “coupled” part H_{ic} of H_i is generated by linear superposition, within the system (H_i, Ω_i) , by all of the modes that appear in the eigenmode decomposition, into modes of Ω_i , of any of the states in $\text{Ran } \Gamma$ ($i = 1$) or $\text{Ran } \Gamma^\dagger$ ($i = 2$).

In the case of pure point spectrum,

$$H_{1c} = \mathcal{O}_{\Omega_1}(\text{Ran } \Gamma) = \bigoplus_{\alpha=1}^n \pi_{1\alpha}(\text{Ran } \Gamma), \tag{102}$$

$$H_{2c} = \mathcal{O}_{\Omega_2}(\text{Ran } \Gamma^\dagger) = \bigoplus_{\alpha=1}^m \pi_{2\alpha}(\text{Ran } \Gamma^\dagger), \tag{103}$$

in which $\pi_{1\alpha}$ and $\pi_{2\alpha}$ are projections onto the eigenspaces of Ω_1 in H_1 and Ω_2 in H_2 , respectively. If Ω_i has continuous spectrum, then, as in our previous discussion, we may replace projections onto the eigenspaces by general spectral projections, that is, those projections that commute with Ω_i (see (95)). The discussion of generalized modes and a construction of the form (97) for H_{ic} is also applicable.

4.2.4. Summary and Theorem

The results of the discussion are collected in the following theorem and proved in Sec. 6 (see the proof of Theorem 14).

We have made the statement about decomposition into modes rigorous in part (i) of the theorem for the case of pure point spectrum, in which the modes are genuine elements of \mathcal{H} ; a weaker rigorous statement in which the modes are replaced by arbitrary spectral projections, is given for general spectrum in part (ii).

Theorem 15 is one of our main results; it summarizes our conclusions about bounding the multiplicity of the frequency operators $\Omega \upharpoonright \mathcal{H}_{\min}$ and $\Omega_{ic} = \Omega_i \upharpoonright H_{ic}$ by the rank of Γ . In particular, the result for Ω_{2c} states that the number of strings needed to construct the minimal extension is bounded by the number of coupling channels.

Theorem 13. *(spectral representation of the minimal extension) Let a conservative system (\mathcal{H}, Ω) be given, and let $(\mathcal{H}_{\min} \subset \mathcal{H}, \Omega \upharpoonright \mathcal{H}_{\min})$ be the minimal conservative extension of the open system $(H_1, \Omega \upharpoonright H_1, a_1(t))$ obtained by projecting the dynamics of (\mathcal{H}, Ω) onto the subspace $H_1 \subset \mathcal{H}$. The coupling operator Γ is assumed to be bounded. (By the “projection” of a subset of a Hilbert space onto a subspace, we refer to the image of the orthogonal projection operator in the Hilbert space onto the subspace.)*

- i. *If Ω has pure point spectrum, then \mathcal{H}_{\min} is the closure of the linear span of the projections of H_1 onto the eigenspaces of Ω in \mathcal{H} :*

$$\mathcal{H}_{\min} = \mathcal{O}_{\Omega}(H_1) = \bigoplus_{\alpha=1}^n \pi_{\alpha}(H_1). \tag{104}$$

If, in addition H_1 contains no nontrivial Ω -invariant subspace, then \mathcal{H}_{\min} is the closure of the linear span of the projections of $\text{Ran } \overset{\circ}{\Gamma}$ onto the eigenspaces of Ω :

$$\mathcal{H}_{\min} = \mathcal{O}_{\Omega}(\text{Ran } \overset{\circ}{\Gamma}) = \bigoplus_{\alpha=1}^n \pi_{\alpha}(\text{Ran } \overset{\circ}{\Gamma}). \tag{105}$$

Here, n may be equal to infinity.

- ii. *\mathcal{H}_{\min} is the closure of the linear span of the projections*

$$\left\{ \pi(H_1) : \pi = \int_{\Delta} dE_{\lambda} \text{ for some interval } \Delta \text{ of } \mathbb{R} \right\}. \tag{106}$$

If H_1 contains no nontrivial Ω -invariant subspace, then \mathcal{H}_{\min} is the closure of the linear span of the projections

$$\left\{ \pi(\text{Ran } \overset{\circ}{\Gamma}) : \pi = \int_{\Delta} dE_{\lambda} \text{ for some interval } \Delta \text{ of } \mathbb{R} \right\}. \tag{107}$$

For arbitrary $\epsilon > 0$, the set of projections can be restricted to those that vanish outside some spectral interval Δ of length ϵ .

- iii. *Let P_1 denote orthogonal projection onto H_1 within \mathcal{H} . If Ω has pure point spectrum, then $\mathcal{H}_{\min} = \mathcal{H}$ if and only if $P_1(\phi) \neq 0$ for each eigenmode ϕ of Ω . If, in addition, H_1 has no nontrivial Ω -invariant subspace, then $\mathcal{H}_{\min} = \mathcal{H}$ if and only if $\overset{\circ}{\Gamma}(\phi) \neq 0$ for each eigenmode ϕ of Ω . In fact,*

$(\mathcal{H} = H_1 \oplus H_2, \Omega)$ is reconstructible if and only if $\overset{\circ}{\Gamma}(\phi) \neq 0$ for each eigenmode ϕ of Ω .

Theorem 14. (spectral representation of the hidden variables) Let the hypotheses of Theorem 13 continue to hold.

- i. If Ω_2 has pure point spectrum, then H_{2c} is the closure of the linear span of the projections of $\text{Ran } \Gamma^\dagger$ onto the eigenspaces of Ω_2 in H_2 :

$$H_{2c} = \mathcal{O}_{\Omega_2}(\text{Ran } \Gamma^\dagger) = \bigoplus_{\alpha=1}^n \pi_{2\alpha}(\text{Ran } \Gamma^\dagger). \tag{108}$$

Here, n may be equal to infinity.

- ii. H_{2c} is the closure of the linear span of the projections

$$\left\{ \pi(\text{Ran } \Gamma^\dagger) : \pi \int_{\Delta} dE_{2,\lambda} \text{ for some interval } \Delta \text{ of } \mathbb{R} \right\}. \tag{109}$$

For arbitrary $\epsilon > 0$, the set of projections can be restricted to those that vanish outside some spectral interval Δ of length ϵ .

- iii. If Ω_2 has pure point spectrum, then $H_{2c} = H_2$ if and only if $\Gamma^\dagger(\phi) \neq 0$ for each eigenmode ϕ of Ω_2 .

The following theorem is a corollary to the preceding theorems. It is one of our main results, which we have alluded to in the introduction and in the construction of the toy model of a solid with frozen degrees of freedom in Sec. 3. The second part shows that the rank of the coupling operator bounds the number of extending strings needed in the minimal conservative extension of an open system.

Theorem 15. (bound on number of strings)

- i. The multiplicity of each spectral value λ of $\Omega \upharpoonright \mathcal{H}_{\min}$ is bounded above by the dimension of H_1 and, if $H_{1c} = H_1$, by twice the rank of Γ :

$$\text{multiplicity}(\lambda) \leq \min\{\dim(H_1), 2 \text{rank}(\Gamma)\}. \tag{110}$$

- ii. The multiplicity of each spectral value λ of $\Omega_i \upharpoonright H_{1c}$ is bounded above by the rank of Γ :

$$\text{multiplicity}(\lambda) \leq \text{rank}(\Gamma). \tag{111}$$

For $i = 2$, this states that the number of abstract strings needed to extend $(H_1, \Omega \upharpoonright H_1, a_1(t))$ minimally to a conservative system is no greater than the number of coupling channels between H_1 and H_2 .

5. DECOMPOSITION OF COUPLED SYSTEMS

It can happen that a given open system splits into two or more smaller independent subsystems, that are decoupled from each other, leading to a natural simplifying decomposition. But, depending on the choice of coordinates, such a natural decomposition may not be evident right away. We ask then if there is a systematic way to find such a decomposition. In this section we intend to answer this question, at least under tractable conditions; the most general conditions are treated in Sec. 6.

Let us return to our observable open system:

$$\partial_t v_1(t) = -i\Omega_1 v_1(t) - \int_0^\infty a_1(\tau) v_1(t - \tau) d\tau + f_1(t) \quad \text{in } H_1, \quad (112)$$

and suppose that there is a subspace of observable variables $H'_1 \subset H_1$ such that an observer confined to this subspace experiences no influence from the rest of the observable space, that is, H'_1 is decoupled from $H''_1 = H_1 \ominus H'_1$ under the dynamics of (112). More precisely, let π'_1 be the orthogonal projection onto H'_1 in H_1 and $\pi''_1 = \mathbb{I}_{H_1} - \pi'_1$ the projection onto H''_1 and let $v'_1(t) = \pi'_1 v_1(t)$ and $f'_1(t) = \pi'_1 f_1(t)$. Then the decoupling of H'_1 means that

$$\pi'_1 \Omega_1 \pi''_1 = 0 \quad \text{and} \quad \pi'_1 a_1(t) \pi''_1 = 0 \quad \text{for all } t, \quad (113)$$

so that $v'(t)$ satisfies a dynamical equation within H'_1 , with no input from H''_1 :

$$\partial_t v'_1(t) = -i\Omega_1 v'_1(t) - \int_0^\infty a_1(\tau) v'_1(t - \tau) d\tau + f'_1(t) \quad \text{in } H'_1. \quad (114)$$

The question then arises: Does this imply the reciprocal condition that that H''_1 evolves independently of H'_1 ? In other words, if H'_1 is not influenced by H''_1 , then does it follow that H''_1 is not influenced by H'_1 ? We shall prove that the answer is affirmative. This means that $\pi''_1 a_1(t) \pi'_1 = 0$ also, so that the splitting $H_1 = H'_1 \oplus H''_1$ is preserved by $a_1(t)$, for all t . In this case, we have a decoupling of the open system (112) into two independent open systems, so that

$$\partial_t v''_1(t) = -i\Omega_1 v''_1(t) - \int_0^\infty a_1(\tau) v''_1(t - \tau) d\tau + f''_1(t) \quad \text{in } H''_1 \quad (115)$$

also holds.

Such decoupling of open systems is easy to understand *from the point of view of the minimal conservative extension* (\mathcal{H}, Ω) of $(H_1, \Omega_1, a_1(t))$. If (\mathcal{H}', Ω') and $(\mathcal{H}'', \Omega'')$ are the minimal conservative extensions of the open systems in H'_1 and H''_1 , then $(\mathcal{H} := \mathcal{H}' \oplus \mathcal{H}'', \Omega := \Omega' \oplus \Omega'')$ is the minimal conservative extension of $(H_1, \Omega_1, a_1(t))$.

The decoupling of the open system (112) is tantamount to the existence of a projection in its conservative extension \mathcal{H} , namely the projection π' onto \mathcal{H}' ,

that commutes both with Ω as well as with the projection P_1 onto H_1 . This is the content of Theorem 17 (and Theorem 21 in Sec. 6) below.

The structure of this decomposition and its implications for the decomposition of Ω_1, Ω_2 , and Γ can be seen in a four-fold decomposition of (\mathcal{H}, Ω) . Denote by $P_2 = \mathbb{I} - P_1$ the projection onto H_2 and by $\pi'' = \mathbb{I} - \pi'$ the projection onto \mathcal{H}' . We note that P_1 and π' commute if and only if \mathcal{H} admits the (orthogonal) decomposition

$$\mathcal{H} = H_1' \oplus H_1'' \oplus H_2' \oplus H_2'', \tag{116}$$

where the components are, respectively, the images of the projections $\pi'P_1, \pi''P_1, \pi'P_2$, and $\pi''P_2$. With respect to the decomposition (116), the operator Ω has the form

$$\Omega = \begin{bmatrix} \Omega_1' & 0 & \Gamma' & 0 \\ 0 & \Omega_1'' & 0 & \Gamma'' \\ \Gamma'^\dagger & 0 & \Omega_2' & 0 \\ 0 & \Gamma''^\dagger & 0 & \Omega_2'' \end{bmatrix}, \tag{117}$$

in which the splittings $H_1 = H_1' \oplus H_1''$ and $H_2 = H_2' \oplus H_2''$ simultaneously diagonalize Ω_1, Ω_2 , and Γ .

This type of system decoupling of a conservative extension (\mathcal{H}, Ω) of a given open system $(H_1, \Omega_1, a_1(t))$ we call *s-invariant* (for system-invariant) with respect to H_1 .

Definition 16. (*s-invariant decomposition*) Let a conservative system (\mathcal{H}, Ω) be given, along with an open subsystem obtained by projecting the dynamics onto a subspace $H_1 \subset \mathcal{H}$, and let P_1, Ω_1, Ω_2 , and Γ be defined as before. A decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$, with projection π' onto \mathcal{H}' , is called *s-invariant* with respect to H_1 (or P_1) if the following equivalent conditions hold:

- i. π' commutes with Ω and P_1 ;
- ii. \mathcal{H}' (or, equivalently, \mathcal{H}'') is of the form $\mathcal{H}' = H_1' \oplus H_2'$, where $H_1' \subset H_1$ and $H_2' \subset H_2$, and \mathcal{H}' is invariant under $\Omega(O_\Omega(\mathcal{H}') = \mathcal{H}')$.

An *s-invariant decomposition* of a conservative extension of $(H_1, \Omega_1, a_1(t))$ is understood to be *s-invariant* with respect to the subspace H_1 .

Theorem 17. (*decoupling criterion*) Let the open system (112) characterized by the triple $(H_1, \Omega_1, a_1(t))$ be given, along with a subspace $H_1' \subset H_1$. The following are equivalent:

- i. The projected dynamics of the open system onto H_1' is not influenced by the dynamics of $H_1 \ominus H_1'$, that is, the dynamical Eq. (114) holds.

ii. The minimal conservative extension (\mathcal{H}, Ω) of the open system admits an s -invariant splitting $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ such that $H'_1 = H_1 \cap \mathcal{H}'$, that is, the blockdiagonal form (117) for Ω holds.

Notice that part (ii) implies that $a_1(t)$ is diagonal with respect to the decomposition $H_1 = H'_1 \oplus H'_2$ so that both (114) and (115) hold. Therefore, by the theorem, (114) (or (115)) is equivalent to (114, 115).

Theorem 18 involves the relation between s -invariant decompositions of conservative systems and the singular values of the coupling operator. We begin with a treatment of an arbitrary countable orthogonal decomposition of H_1 and H_2 that is invariant under the internal actions given by Ω_1 and Ω_2 but does not necessarily correspond to an s -invariant decomposition:

$$H_i = \bigoplus_{\alpha=1}^{n_i} H_{i\alpha}, \quad \mathbb{I}_{H_i} = \sum_{\alpha=1}^{n_i} \pi_{i\alpha}, \quad i = 1, 2, \tag{118}$$

where $\pi_{i\alpha}$, $i = 1, 2$, are the orthogonal projections onto the the subspaces $H_{i\alpha}$ and the n_i are allowed to be ∞ .

The frequency and coupling operators split as follows:

$$\Omega_i = \sum_{\alpha=1}^{n_i} \Omega_{i\alpha}, \tag{119}$$

$$\Gamma = \mathbb{I}_{H_1} \Gamma \mathbb{I}_{H_2} = \sum_{\alpha=1}^{n_1} \sum_{\beta=1}^{n_2} \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} = \pi_{1\alpha} \Gamma \pi_{2\beta}. \tag{120}$$

Given $\Gamma_{\alpha\beta} = \pi_{1\alpha} \Gamma \pi_{2\beta}$, we have also $\Gamma_{\alpha\beta}^\dagger = \pi_{2\beta} \Gamma^\dagger \pi_{1\alpha}$. In block-matrix form, with $n_1 = 2$ and $n_2 = 3$, Ω has the form

$$\Omega = \begin{bmatrix} \Omega_1 & \Gamma \\ \Gamma^\dagger & \Omega_2 \end{bmatrix} = \begin{bmatrix} \Omega_{11} & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ 0 & \Omega_{12} & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{11}^\dagger & \Gamma_{21}^\dagger & \Omega_{21} & 0 & 0 \\ \Gamma_{12}^\dagger & \Gamma_{22}^\dagger & 0 & \Omega_{22} & 0 \\ \Gamma_{13}^\dagger & \Gamma_{23}^\dagger & 0 & 0 & \Omega_{23} \end{bmatrix}. \tag{121}$$

We refine our decomposition of Γ through its singular-value decomposition, aided by the Hilbert-space isomorphism between the range of Γ in H_1 and the range of Γ^\dagger in H_2 , described in Sec. 2 (p. 11):

$$U : \text{Ran } \Gamma \rightarrow \text{Ran } \Gamma^\dagger. \tag{122}$$

For the sake of technical simplicity we restrict discussion in this work to the situation in which the spectrum of $\Gamma_R \Gamma_R^\dagger$ consists only of eigenvalues and their

accumulation points, and state the result, Theorem 18, for this case. Its proof as well as a partial generalization of it is given in Sec. 6 (Theorem 22). The eigenspaces of $\Gamma_R \Gamma_R^\dagger$ and $\Gamma_R^\dagger \Gamma_R$ for the same eigenvalue are identified isometrically through U . Let $r = \text{rank } \Gamma = \text{rank } \Gamma^\dagger$, which is allowed to be ∞ , and let $\{g_q\}_{q=1}^r$ be an orthonormal basis for $\text{Ran } \Gamma$ consisting of eigenvectors of $\Gamma_R \Gamma_R^\dagger$ with eigenvalues $\gamma_q > 0$, and put $g'_q = U g_q$. Each of the vectors g_q and g'_q is an eigenvector (or generalized eigenvector if Γ is unbounded) of respectively $\Gamma_R \Gamma_R^\dagger$ and $\Gamma_R^\dagger \Gamma_R$.

With the help of the Dirac notation, in which $|g_q\rangle$ indicates the vector g_q and $\langle g_q|$ the linear functional of projection onto g_q , we can write $\Gamma \Gamma^\dagger$ and its adjoint as a sum of rank-one operators:

$$\Gamma \Gamma^\dagger = \sum_{q=1}^r \gamma_q |g_q\rangle \langle g_q|, \quad \Gamma^\dagger \Gamma = \sum_{q=1}^r \gamma_q |g'_q\rangle \langle g'_q|. \tag{123}$$

(If $|g_q\rangle$ is a genuine eigenvector of $\Gamma_R \Gamma_R^\dagger$ as we assumed, then $|g_q\rangle \in H_1$; if it is generalized, then $|g_q\rangle \in [H_1]_-$, where $[H_1]_+ \subset H_1 \subset [H_1]_-$ is a proper furnishing of H_1). If all γ_q are different, then the representations (123) are unique. If some of the γ_q coincide, then they are not, and we may choose orthonormal eigenvectors arbitrarily from the eigenspace. With this structure, Γ_R can be written as a sum of linearly independent rank-one operators:

$$\Gamma_R = \sum_{q=1}^r \Gamma_q, \quad \Gamma_q = \sqrt{\gamma_q} |g_q\rangle \langle g'_q|, \quad \gamma_q > 0, \tag{124}$$

$$\langle g_p | |g_q\rangle = \langle g'_p | |g'_q\rangle = \delta_{pq}. \tag{125}$$

For each $q = 1, \dots, r$, decompose $|g_q\rangle$ and $|g'_q\rangle$ with respect to the Hilbert space decompositions (118):

$$|g_q\rangle = \sum_{\alpha=1}^{n_1} |g_q^\alpha\rangle, \quad |g'_q\rangle = \sum_{\alpha=1}^{n_2} |g_q'^\alpha\rangle. \tag{126}$$

It follows that

$$\Gamma_q = \sqrt{\gamma_q} \sum_{\alpha=1}^{n_1} \sum_{\beta=1}^{n_2} |g_q^\alpha\rangle \langle g_q'^\beta|, \tag{127}$$

so that Γ is decomposed as

$$\Gamma = \sum_{\alpha=1}^{n_1} \sum_{\beta=1}^{n_2} \Gamma_{\alpha\beta}, \quad \Gamma_{\alpha\beta} = \sum_{q=1}^r \sqrt{\gamma_q} |g_q^\alpha\rangle \langle g_q'^\beta|. \tag{128}$$

This decomposition shows explicitly the coupling between the components of H_1 and the components of H_2 in terms of the spectral structure of the coupling

operator. $H_{1\alpha}$ is directly coupled with $H_{2\beta}$ if and only if $\Gamma_{\alpha\beta} = 0$. We note, however, that, for a fixed pair (α, β) , the rank-one operators $|g_q^\alpha\rangle\langle g_q'^\beta|$, for $q = 1, \dots, r$, are not in general independent, so that $\Gamma_{\alpha\beta}$ may be zero even if, for some q , g_q^α and $g_q'^\beta$ are both nonzero; in fact, the cardinality of $\{q : g_q^\alpha\langle g_q'^\beta| \neq 0\}$ may exceed the rank of $\Gamma_{\alpha\beta}$.

We organize the coupling information by introducing the $n_1 \times n_2$ coupling matrix M_Γ with entries

$$[M_\Gamma]_{\alpha\beta} = \text{rank } \Gamma_{\alpha\beta}. \tag{129}$$

The $\alpha\beta$ -component of the coupling matrix can be thought of as the number of coupling channels between the components $H_{1\alpha}$ and $H_{2\beta}$. Rows of M_Γ containing all zeros indicate components of H_1 that split from the rest of the system $H_1 \oplus H_2$, and columns of all zeros indicate components of H_2 that split from the rest of the system. If the subspaces $H_{i\alpha}$ can be reordered in such a way that M_Γ attains a diagonal block form (with not necessarily square blocks), then we see that the system splits into completely decoupled subsystems, each with a nontrivial component in each of H_1 and H_2 made up of components $H_{i\alpha}(i = 1, 2)$. *This leads to an s-invariant decomposition.* In this case, it is possible to choose the g_q differently if necessary so that, for $\alpha\beta$ off of the diagonal blocks, we have $|g_q^\alpha\rangle\langle g_q'^\beta| = 0$ for all q , as we will see.

We now examine s-invariant decompositions in more detail, that is, how $\mathcal{H} = H_1 \oplus H_2$ can be decomposed into independently evolving components of the form $H_{1\alpha} \oplus H_{2\alpha}$:

$$\mathcal{H} = \bigoplus_{\alpha=1}^n (H_{1\alpha} \oplus H_{2\alpha}), \tag{130}$$

where $H_{1\alpha} \oplus H_{2\alpha}$ is invariant under Ω for each α . This means that $H_{i\alpha}$ is invariant under Ω_i for $i = 1, 2$ and $\alpha = 1, \dots, n$ and that the coupling operators $\Gamma_{\alpha\beta}$ are equal to zero for $\alpha \neq \beta$. In other words, *this decomposition simultaneously block-diagonalizes $\Omega_1, \Omega_2,$ and Γ .* For $n = 2$, for example, Ω has the form

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & \Gamma_{11} & 0 \\ 0 & \Omega_{12} & 0 & \Gamma_{22} \\ \Gamma_{11}^\dagger & 0 & \Omega_{21} & 0 \\ 0 & \Gamma_{22}^\dagger & 0 & \Omega_{22} \end{bmatrix} \tag{131}$$

and $\Gamma\Gamma^\dagger$ has the block-diagonal form

$$\Gamma\Gamma^\dagger = \begin{bmatrix} \Gamma_{11}\Gamma_{11}^\dagger & 0 \\ 0 & \Gamma_{22}\Gamma_{22}^\dagger \end{bmatrix}, \tag{132}$$

from which we see that any eigenvector of $\Gamma\Gamma^\dagger$ is decomposed with respect to $H_1 = \bigoplus_{\alpha=1}^n H_{1\alpha}$ into a sum of eigenvectors (possibly zero) of $\Gamma\Gamma^\dagger$ with the same eigenvalue.

Thus each eigenspace of $\Gamma\Gamma^\dagger$ admits an orthogonal decomposition into its intersections with all of the $H_{1\alpha}$. This is what allows us to choose the basis $\{g_q\}$ so that each is contained in one of the $H_{1\alpha}$. It follows that g'_q is in $H_{2\alpha}$.

We then ask, given any choice of basis $\{g_q\}$, what is the finest decomposition of the form (130) such that each g_q is contained in one of the $H_{1\alpha}$? The answer requires considering the orbits of the vectors g_q under Ω_1 and the orbits of the vectors g'_q under Ω_2 . We see that if $g_q \in H_{1\alpha}$ for some q and α , it is required by the invariance of $H_{i\alpha}$ under Ω_i , for $i = 1, 2$, that $\mathcal{O}_{\Omega_1}(\{g_q\}) \in H_{1\alpha}$ and $\mathcal{O}_{\Omega_2}(\{g'_q\}) \in H_{2\alpha}$. Further, the orbit $\mathcal{O}_{\Omega_1}(\{g_q\})$ must be orthogonal to every g_p that is not in $H_{1\alpha}$ and the orbit $\mathcal{O}_{\Omega_2}(\{g'_q\})$ must be orthogonal to every g'_p that is not in $H_{2\alpha}$.

Theorem 18. (canonical decomposition) Assume that the spectrum of $\Gamma\Gamma^\dagger$ consists of a countable set of eigenvalues (and their accumulation points).

i. Let an s -invariant splitting of (\mathcal{H}, Ω) be given:

$$\mathcal{H} = \bigoplus_{\alpha=1}^n (H_{1\alpha} \oplus H_{2\alpha}). \tag{133}$$

Then there exists an orthonormal Hilbert-space basis $\{g_q\}_{q=1}^r$ for $\text{Ran } \Gamma$ of eigenvectors of $\Gamma\Gamma^\dagger$ and corresponding basis $\{g'_q = U g_q\}$ for $\text{Ran } \Gamma^\dagger$ such that for each q , there exists α such that $g_q \in H_{1\alpha}$ and $g'_q \in H_{2\alpha}$.

ii. Given an arbitrary choice of basis $\{g_q\}_{q=1}^r$ for $\text{Ran } \Gamma$ consisting of eigenvectors of $\Gamma\Gamma^\dagger$, it follows that the finest s -invariant splitting (of the form (133)) such that each g_q is in some $H_{1\alpha}$ is obtained from the orbits

$$H_{1\alpha} = \mathcal{O}_{\Omega_1}(\{g_q : q \in V_\alpha\}) \quad \text{and} \quad H_{2\alpha} = \mathcal{O}_{\Omega_2}(\{g'_q : q \in V_\alpha\}), \tag{134}$$

in which the V_α are the minimal (disjoint) subsets of $\{1, \dots, r\}$ such that $\mathcal{O}_{\Omega_1}(\{g_q : q \in V_\alpha\}) \perp g_p$ and $\mathcal{O}_{\Omega_2}(\{g'_q : q \in V_\alpha\}) \perp g'_p$ for all $p \notin V_\alpha$.

In part (ii) it is tacitly implied that such minimal subsets are well defined.

6. PROOFS OF THEOREMS

In this section we formulate detailed statements on the structure of open systems and provide their proofs, which encompass the proofs of the theorems from Secs. 4 and 5. The development follows that of those sections.

We use the same notation as in the previous sections. We are given a conservative system (\mathcal{H}, Ω) :

$$\partial_t \mathcal{V} = -i \Omega \mathcal{V} + \mathcal{F}, \tag{135}$$

and an orthogonal splitting of the Hilbert space

$$\mathcal{H} = H_1 \oplus H_2, \tag{136}$$

with respect to which Ω has the form

$$\Omega = \begin{bmatrix} \Omega_1 & \Gamma \\ \Gamma^\dagger & \Omega_2 \end{bmatrix}, \tag{137}$$

and we let P_1 denote projection onto H_1 and define, as before,

$$\overset{\circ}{\Omega} = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix}, \quad \overset{\circ}{\Gamma} = \begin{bmatrix} 0 & \Gamma \\ \Gamma^\dagger & 0 \end{bmatrix}, \quad \Omega = \overset{\circ}{\Omega} + \overset{\circ}{\Gamma}, \tag{138}$$

As $\text{Ran } \Gamma$ and $\text{Ran } \Gamma^\dagger$ are isomorphic through the isomorphism U (31), we may let H_0 be a standard Hilbert space on which the operator Γ_R is represented by a self-adjoint operator Γ_0 . This means that there are unitary operators

$$U_1 : H_0 \rightarrow \text{Ran } \Gamma \subseteq H_1, \quad U_2 : H_0 \rightarrow \text{Ran } \Gamma^\dagger \subseteq H_2 \tag{139}$$

with

$$U = U_2 U_1^{-1} : \text{Ran } \Gamma \rightarrow \text{Ran } \Gamma^\dagger \tag{140}$$

such that

$$\Gamma_0 = U_2^{-1} \Gamma^\dagger U_1 = U_1^{-1} \Gamma U_2. \tag{141}$$

Since $\text{Null } \Gamma^\dagger \perp \text{Ran } \Gamma$, Γ^\dagger is completely determined by its action on $\text{Ran } \Gamma$, and the positive operator $\Gamma_R \Gamma_R^\dagger$ restricted to $\text{Ran } \Gamma$ is represented by Γ_0^2 on H_0 through the isometric isomorphism given by U_1 ; the analogous structure holds for $\Gamma_R^\dagger \Gamma_R$:

$$\Gamma_R \Gamma_R^\dagger = U_1 \Gamma_0^2 U_1^{-1}, \quad \Gamma_R^\dagger \Gamma_R = U_2 \Gamma_0^2 U_2^{-1}. \tag{142}$$

6.1. Reconstructibility from Open Subsystems

The following statement is a detailed version of Theorem 9.

Theorem 19. *(system reconstructibility) There exists a unique minimal sub-Hilbert-space \mathcal{H}' of \mathcal{H} with the following properties:*

- a. \mathcal{H}' is Ω -invariant ($\mathcal{O}_\Omega(\mathcal{H}') = \mathcal{H}'$) and, hence, $(\mathcal{H}', \Omega \upharpoonright \mathcal{H}')$ is conservative;

- b. \mathcal{H}' is P_1 -invariant, that is, $\mathcal{H}' = H_{1c} \oplus H_{2c}$, where $H_{1c} \subseteq H_1$ and $H_{2c} \subseteq H_2$;
- c. $\text{Ran } \Gamma \subseteq \mathcal{H}'$.

Let $\Omega_{ic} = \Omega_i \upharpoonright H_{ic}$ for $i = 1, 2$, $\Gamma_c = \Gamma \upharpoonright H_{2c}$, $a_1(t) = \Gamma e^{-i\Omega_2 t} \Gamma^\dagger$, and $a_2(t) = \Gamma^\dagger e^{-i\Omega_1 t} \Gamma$. The the following hold

- i. $H_{1c} = \mathcal{O}(\text{Ran } \Gamma)$ and $H_{2c} = \mathcal{O}_{\Omega_2}(\text{Ran } \Gamma^\dagger)$.
- ii. $(\mathcal{H}' = H_{1c} \oplus H_{2c}, \Omega \upharpoonright \mathcal{H}')$ is reconstructible, $H_1 \oplus H_{2c}$ is the unique minimal conservative extension of $(H_1, \Omega_1 a_1(t))$ contained in \mathcal{H} , and $H_{1c} \oplus H_2$ is the unique minimal conservative extension of $(H_2, \Omega_2, a_2(t))$ contained in \mathcal{H} .
- iii. $a_1(t) \upharpoonright H_{1c} = \Gamma_c e^{-i\Omega_{2c} t} \Gamma_c^\dagger$ and $a_1(t) \upharpoonright (H_1 \ominus H_{1c}) = 0$,
 $a_2(t) \upharpoonright H_{2c} = \Gamma_c^\dagger e^{-i\Omega_{2c} t} \Gamma_c$ and $a_2(t) \upharpoonright (H_2 \ominus H_{2c}) = 0$.
 Thus the system $H_{1c} \oplus H_{2c}$ completely determines the friction functions $a_1(t)$ and $a_2(t)$. Neither $a_1(t)$ nor $a_2(t)$ is determined by any proper Ω -invariant subsystem of $H_{1c} \oplus H_{2c}$ of the form $\tilde{H}_1 \oplus \tilde{H}_2$, where $\tilde{H}_i \subset H_i, i = 1, 2$.
- iv. $(H_1 \oplus H_2, \Omega)$ is reconstructible if and only if H_1 and H_2 have no nontrivial Ω -invariant subspaces.

Proof: There exists a sub-Hilbert-space \mathcal{H}' of \mathcal{H} possessing properties (a–c) because \mathcal{H} is such a subspace. Let \mathcal{H}' be an arbitrary such space. First, we show that $\text{Ran } \Gamma^\dagger \subset \mathcal{H}'$. We have

$$\text{Ran } \Gamma^\dagger = \Gamma^\dagger(H_1) = \Gamma^\dagger(\text{Ran } \Gamma) \tag{143}$$

since $\text{Null } \Gamma^\dagger = H_1 \ominus \overline{\text{Ran } \Gamma}$. Let $w \in \Gamma^\dagger(\text{Ran } \Gamma)$, say $w = \Gamma^\dagger(u)$ for some $u \in \text{Ran}(\Gamma) \subset \mathcal{H}' \cap H_1$. Let $\epsilon > 0$ be given. Since Ω_1 is densely defined in $H_1 \cap \mathcal{H}'$, there exists $v \in \text{Dom } \Omega_1 \cap \mathcal{H}'$ such that $\|u - v\| < \epsilon / \|\Gamma^\dagger\|$ and hence $\|w - \Gamma^\dagger(v)\| = \|\Gamma^\dagger(u - v)\| < \epsilon$. By (a, b), we obtain

$$\Gamma^\dagger(v) = (I - P_1)(\Omega_1(v) + \Gamma^\dagger(v)) = (I - P_1)\Omega(v) \in \mathcal{H}'. \tag{144}$$

Since ϵ is arbitrary, we conclude that $w \in \mathcal{H}'$. We now know that

$$\mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma}) \subseteq \mathcal{H}'. \tag{145}$$

But, using the definition (22) of the orbit under the action of two operators, we see that $\mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma})$ is itself P_1 -invariant as follows:

$$\mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\Omega, \overset{\circ}{\Gamma}}(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\overset{\circ}{\Omega}, \overset{\circ}{\Gamma}}(\text{Ran } \overset{\circ}{\Gamma}) = \mathcal{O}_{\overset{\circ}{\Omega}}(\text{Ran } \overset{\circ}{\Gamma}) \tag{146}$$

$$= \mathcal{O}_{\overset{\circ}{\Omega}}(\text{Ran } \Gamma \oplus \text{Ran } \Gamma^\dagger) = \mathcal{O}_{\Omega_1}(\text{Ran } \Gamma) \oplus \mathcal{O}_{\Omega_2}(\text{Ran } \Gamma^\dagger). \tag{147}$$

Defining H_{1c} and H_{2c} as in (i), we see that $H_{1c} \oplus H_{2c}$ both satisfies properties (a–c) and is contained in our arbitrarily chosen \mathcal{H}' with these properties. This proves the uniqueness of a minimal subspace satisfying (a–c), namely, $\mathcal{H}' = \mathcal{O}_\Omega(\text{Ran } \overset{\circ}{\Gamma}) = H_{1c} \oplus H_{2c}$, as well as property (i).

To prove that $(\mathcal{H}', \Omega \upharpoonright \mathcal{H}')$ is reconstructible (part (ii)), we must show that $\mathcal{H}' = \mathcal{O}_\Omega(H_{1c})$, which is the state space of the minimal conservative extension for H_{1c} in \mathcal{H} , and that $\mathcal{H}' = \mathcal{O}_\Omega(H_{2c})$. Observe that $\text{Ran } \Gamma \in \mathcal{H}'$ and choose again w and v as before. We have $\Omega_1(v) \in H_{1c}$, so that

$$\Gamma^\dagger(v) = \Omega(v) - \Omega_1(v) \in \mathcal{O}_\Omega(H_{1c}), \tag{148}$$

and $\|\Gamma^\dagger(v) - w\| < \epsilon$. We conclude that

$$\text{Ran } \overset{\circ}{\Gamma} \subseteq \mathcal{O}_\Omega(H_{1c}). \tag{149}$$

But since $\mathcal{H}' = \mathcal{O}_\Omega(\text{Ran } \Gamma^\circ)$ and $H_{1c} \subseteq \mathcal{H}'$, we obtain

$$\mathcal{O}_\Omega(H_{1c}) = \mathcal{H}'. \tag{150}$$

An analogous argument applies to H_{2c} .

To prove the rest of part (ii), define $H_{1d} = H_1 \ominus H_{1c}$ and $H_{2d} = H_2 \ominus H_{2c}$. Since $H_{1d} \oplus H_{2d} = \mathcal{H} \ominus \mathcal{H}'$, $H_{1d} \oplus H_{2d}$ is Ω -invariant. Since H_{1d} is perpendicular to $\text{Ran } \Gamma$, for $v \in H_{1d} \cap \text{Ran } \Omega$, $\Omega(v) = \Omega_1(v) + \Gamma^\dagger(v) = \Omega_1(v) \in H_{1d}$, and we see that H_{1d} itself is Ω -invariant. Therefore, so is $H_1 \oplus H_{2c}$, and we obtain

$$\mathcal{O}_\Omega(H_1) = \mathcal{O}_\Omega(H_{1d}) \oplus \mathcal{O}_\Omega(H_{1c}) = H_1 \oplus H_{2c} \tag{151}$$

That $\mathcal{O}_\Omega(H_2) = H_2 \oplus H_{1d}$ is shown similarly.

To prove part (iii), let $v \in H_{1c}$ be given. Then

$$\begin{aligned} a_1(t)v &= \Gamma e^{-i\Omega_2 t} \Gamma^\dagger v = \Gamma e^{-i\Omega_2 t} \Gamma_c^\dagger v \quad (\text{because } v \in H_{1c}) \\ &= \Gamma e^{-i\Omega_{2c} t} \Gamma_c^\dagger v \quad (\text{because } \Gamma_c^\dagger v \in \text{Ran } \Gamma^\dagger \subset H_{2c}) \\ &= \Gamma_c e^{-i\Omega_{2c} t} \Gamma_c^\dagger v \quad (\text{because } e^{-i\Omega_{2c} t} \Gamma_c^\dagger v \in H_{2c}). \end{aligned}$$

Let $v \in H_1 \ominus H_{1c}$ be given. Then $v \perp \text{Ran } \Gamma$, so that $v \in \text{Ker } \Gamma^\dagger$. The analogous statement about $a_2(t)$ is proven similarly. Finally, if $\tilde{H}_1 \oplus \tilde{H}_2$ is a proper subspace of $\mathcal{H}' = H_{1c} \oplus H_{2c}$ that is invariant under Ω and P_1 , then $\text{Ran } \Gamma \not\subseteq \tilde{H}_1$. This is because \mathcal{H}' is the minimal such space that contains $\text{Ran } \Gamma$. Since $a_1(0) = \Gamma \Gamma^\dagger$, we have $\text{Ran } a_1(0) = \text{Ran } \Gamma \not\subseteq \tilde{H}_1$. This means that the restriction of the system $(\mathcal{H}', \Omega \upharpoonright \mathcal{H}')$ to $\tilde{H}_1 \oplus \tilde{H}_2$ does not determine $a_1(0)$, and therefore does not determine the function $a_1(t)$.

To prove part (iv), first observe that, if $H_1 \oplus H_2$ is not reconstructible, then H_{1d} , which is Ω -invariant, is nontrivial. Conversely, suppose that H_1 has an Ω -invariant subspace $H'_1 \subset H_1$. Set $H''_1 = H_1 \ominus H'_1$, and let $H''_1 \oplus H''_2 = \mathcal{O}_\Omega(H''_1)$. Then the minimal conservative extension of $(H_1, \Omega_1, a_1(t))$ is $\mathcal{O}_\Omega(H_1) =$

$H'_1 \oplus H''_1 \oplus H''_2$, so that $H''_2 = H_{2c}$. For all $v \in H_{2c} \cap \text{Dom}(\Omega)$, $\Omega(v) = \Gamma(v) + \Omega_2(v) \in H''_1 \oplus H_{2c}$. Thus $\Gamma(v) \perp H'_1$, so that $\text{Ran } \Gamma \perp H'_1$ and hence $\text{Ran } \Gamma \subset H''_1$. Since $H''_1 \oplus H_{2c} = \mathcal{O}_\Omega(H''_1 \oplus H_{2c}) \subset \mathcal{O}_\Omega(\text{Ran } \Gamma^\circ) = H_{1c} \oplus H_{2c}$, we obtain $H'_1 \perp H_{1c}$ so that $H_1 \oplus H_2$ is not reconstructible. \square

Remark 20. *Of all Ω -invariant subsystems of the form $\tilde{H}_1 \oplus \tilde{H}_2$ with $\tilde{H}_i \subset H_i, i = 1, 2, H_{1c} \oplus H_{2c}$ is the minimal reconstructible one that has the property that $\text{Ran } \Gamma \subset H_{1c}$. There may exist Ω -invariant subsystems of the same form such that $\text{Ran } \Gamma \not\subset H_{1c}$ (this is dealt with in Theorem 18) and Ω -invariant subsystems that are not of this form, which do not concern us.*

Theorems 13, 14, and 15 are rather straightforward applications of standard spectral theory of self-adjoint operators in separable Hilbert space. We shall set down the general framework and prove those results. The relevant material can be found, for example, in Akhiezer and Glazman (Ref. 1) or (Ref. 9).

Let dE_λ be the spectral resolution of the identity for a self-adjoint operator Ω in the Hilbert space \mathcal{H} . This means that dE_λ is an (orthogonal) projection-valued Borel measure on \mathbb{R} such that, for each $v \in \mathcal{H}$, the vector-valued function of μ given by

$$\int_{(-\infty, \mu]} d(E_\lambda v) \tag{152}$$

is right-continuous,

$$\int_{\mathbb{R}} d(E_\lambda v) = \lim_{\mu \rightarrow \infty} \int_{(-\infty, \mu]} d(E_\lambda v) = v, \tag{153}$$

and, for each $f \in C_c(\mathbb{R})$,

$$f(\Omega)v = \int_{\mathbb{R}} f(\lambda)d(E_\lambda v); \tag{154}$$

integration is understood in the Lebesgue–Stieltjes sense.

The orbit of a subset $S \subset \mathcal{H}$ generated by the action of Ω can be expressed in terms of continuous functions of Ω or in terms of spectral projections:

$$\mathcal{O}_\Omega(S) = \text{closure of span } \{f(\Omega)v : v \in S, f \in C_c(\mathbb{R})\} \tag{155}$$

$$= \text{closure of span } \left\{ \int_B d(E_\lambda v) : v \in S, B \text{ a Borel set} \right\} \tag{156}$$

$$= \text{closure of span } \left\{ \int_\Delta d(E_\lambda v) : v \in S, \Delta \text{ an interval in } \mathbb{R} \text{ with } |\Delta| < d \right\}, \tag{157}$$

in which ϵ is an arbitrary positive real number.

Proof of Theorems 13, 14, and 15. Parts (ii) of Theorems 13 and 14 are statements of the representation (157) of the Ω -orbits of H_1 and $\text{Ran } \overset{\circ}{\Gamma}$ and the Ω_2 -orbit of $\text{Ran } \Gamma^\dagger$. We have already shown (see Eq. 101) that \mathcal{H}_{\min} is generated through the action of Ω on the range of $\overset{\circ}{\Gamma}$.

To prove parts (i) of these theorems, observe that, in the case of pure point spectrum,

$$dE_\lambda = \sum_{j=1}^N E_j \delta(\lambda_j - \lambda), \tag{158}$$

in which δ is the unit measure concentrated at $\lambda = 0$, the λ_j are the distinct eigenvalues of Ω , the E_j are orthogonal projections, and N may be equal to ∞ . Consider the representation (156): for any vector $v \in S$ and Borel set B ,

$$\int_B d(E_\lambda v) = \sum_{j:\lambda_j \in B} E_j v. \tag{159}$$

In particular, $E_j v \in \mathcal{O}_\Omega$ (by taking $B = \{\lambda_j\}$) for $j = 1, \dots, N$, and each vector $\int_B d(E_\lambda v)$ is in the closure of the linear span of the $E_j v$. It follows that

$$\mathcal{O}_\Omega(S) = \text{closure of span } \{E_j v : j \in \{1, \dots, N\}, v \in S\} = \bigoplus_{i=1}^N E_j(S). \tag{160}$$

The statements (i) of the theorems follow from applying this result to the respective operator Ω and set S .

To prove parts (iii) of the theorems, observe that, since $\mathcal{H} = \bigoplus_{i=1}^N \text{Ran}(E_j)$,

$$\mathcal{O}_\Omega(S)^\perp = \bigoplus_{i=1}^N (\text{Ran}(E_j) \ominus E_j(S)). \tag{161}$$

Suppose that S is a subspace of \mathcal{H} , and let P denote the projection onto S . Then, for any eigenvector, say $\phi \in \text{Ran}(E_j)$, we have a splitting $\phi = \phi_1 + \phi_2$, where $\phi_1 \in E_j(S) \subseteq \mathcal{O}_\Omega(S)$ and $\phi_2 \in \text{Ran}(E_j) \ominus E_j(S) \subseteq \mathcal{O}_\Omega(S)^\perp$. Thus, $P(\phi_1) = 0$ and $\phi_2 = E_j(v)$ for some $v \in S$. From

$$\|\phi_2\|^2 = \|E_j(v)\|^2 = \langle v | E_j(v) \rangle = \langle v | \phi_2 \rangle, \tag{162}$$

we infer that $\phi_2 = 0$ if and only if $\phi_2 \perp S$, that is, if and only if $P(\phi_2) = 0$. But we also see that $\phi_2 = 0$ if and only if $\phi \perp E_j(S)$, which is true if and only if $\phi \in \mathcal{O}_\Omega(S)^\perp$. It follows that

$$\mathcal{O}_\Omega(S)^\perp = \{0\} \text{ if and only if } P(\phi) \neq 0 \text{ for each (nonzero) eigenvector of } \Omega. \tag{163}$$

This result applies directly to the first part of (iii) of Theorem 13 because $\mathcal{H}_{\min} = \mathcal{O}_\Omega(H_1)$. For the second part and part (iii) of Theorem 14, (163) applies

after observing that (1) $\mathcal{H}_{\min} = H_1 \oplus H_{2c} = H_{1c} \oplus H_{2c} = \mathcal{O}_{\Omega}(\text{Ran } \overset{\circ}{\Gamma})$ if and only if $H_{1c} = H_1$ and (2) P denotes projection onto $\text{Ran } \overset{\circ}{\Gamma}$, then for each $v \in \mathcal{H}$, $P(v) = 0$ if and only if $\Gamma(v) = 0$ —this is because $\overset{\circ}{\Gamma}$ is self-adjoint so that the nullspace of $\overset{\circ}{\Gamma}$ is equal to $(\text{Ran } \overset{\circ}{\Gamma})^{\perp}$.

Theorem 15 follows from the fact that, if S is a subspace of \mathcal{H} , then the multiplicity of Ω restricted to $\mathcal{O}_{\Omega}(S)$ is bounded by the dimension of S . \square

6.2. Decomposition of Coupled Systems

The next statement on the equivalence between decoupling of an open system and s -invariant decompositions of its minimal conservative extension is a detailed version of Theorem 17.

Theorem 21. *(decoupling and s -invariant decomposition) Let an open linear system $(H_1, \Omega_1, a_1(t))$ be given, and let (\mathcal{H}, Ω) be its minimal conservative extension, with $P_1, H_2 = \mathcal{H} \ominus H_1, \Omega_2$, and $\Gamma : H_2 \rightarrow H_1$ defined as before.*

- i. *Let $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ be an s -invariant decomposition, with $H_1 = H'_1 \oplus H''_1$, where $H'_1 = P_1(\mathcal{H}')$. Then the open system $(H_1, \Omega_1, a_1(t))$ is decoupled, that is, if π'_1 and π''_1 are projections in H_1 onto H'_1 and H''_1 , then $\pi'_1 a_1(t) \pi''_1 = 0$ and $\pi''_1 a_1(t) \pi'_1 = 0$. Equivalently, putting $v'_1(t) = \pi'_1 v(t)$, $a'_1(t) = \pi'_1 a_1(t) \pi'_1$ and $f'_1(t) = \pi'_1 f(t)$, the dynamics of the open system $(H_1, \Omega_1, a_1(t))$ are decoupled into*

$$\partial_t v'_1(t) = -i\Omega_1 v'_1(t) - \int_0^{\infty} a'_1(\tau) v'_1(t - \tau) d\tau + f'_1(t) \tag{164}$$

and

$$\partial_t v''_1(t) = -i\Omega_1 v''_1(t) - \int_0^{\infty} a''_1(\tau) v''_1(t - \tau) d\tau + f''_1(t). \tag{165}$$

- ii. *Let $H_1 = H'_1 \oplus H''_1$ be an Ω -invariant decomposition with corresponding projections π'_1 and π''_1 , and suppose that $\pi'_1 a_1(t) \pi''_1 = 0$, that is, that (164) holds (the evolution of H'_1 is not influenced by H''_1). Then there exists an s -invariant decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$, with projection π' onto \mathcal{H}' , such that $H'_1 = \text{Ran } \pi' P_1$. In addition, $(\mathcal{H}', \Omega \upharpoonright \mathcal{H}')$ is the minimal conservative extension of $(H'_1, \Omega_1 \upharpoonright H'_1, \pi'_1 a_1(t) \pi'_1)$.*

Proof: The proof of the first part is straightforward. To prove the second statement, let

$$a'_1(t) = \pi'_1 a_1(t) \pi'_1, \quad a''_1(t) = \pi''_1 a_1(t) \pi''_1, \quad \tilde{a}''_1(t) = \pi''_1 a_1(t) \pi'_1, \tag{166}$$

so that, in block-matrix form with respect to the decomposition $H_1 = H'_1 \oplus H''_1$, $a_1(t)$ has the representation

$$a_1(t) = \begin{bmatrix} a'_1(t) & 0 \\ \tilde{a}''_1(t) & a''_1(t) \end{bmatrix}. \tag{167}$$

Since $a_1(0) = \Gamma \Gamma^\dagger$ is self-adjoint, we have

$$\tilde{a}''_1(0) = \pi'_1 a_1(0) \pi'_1 = (\pi'_1 a_1(0) \pi'_1)^\dagger = 0, \tag{168}$$

or, in matrix form,

$$\Gamma \Gamma^\dagger = a_1(0) = \begin{bmatrix} a'_1(0) & 0 \\ 0 & a''_1(0) \end{bmatrix}. \tag{169}$$

This form gives rise to a splitting of $\text{Ran } \Gamma$ that is invariant under $\Gamma_R \Gamma_R^\dagger$:

$$\text{Ran } \Gamma = \text{Ran } a'_1(0) \oplus \text{Ran } a''_1(0) = \pi'_1 \text{Ran } \Gamma \oplus \pi''_1 \text{Ran } \Gamma = U_1(H'_0) \oplus U_1(H''_0), \tag{170}$$

in which

$$H_0 = H'_0 \oplus H''_0 = U_1^{-1} \text{Ran } a'_1(0) \oplus U_1^{-1} \text{Ran } a''_1(0) \tag{171}$$

is the induced Γ_0 -invariant splitting of the standard Hilbert space H_0 for Γ (by virtue of $\Gamma_R \Gamma_R^\dagger = U_1 \Gamma_0^\dagger U_1^{-1}$). This gives a $\Gamma_R^\dagger \Gamma_R$ -invariant splitting of $\text{Ran } \Gamma^\dagger$:

$$\text{Ran } \Gamma^\dagger = U_2(H'_0) \oplus U_2(H''_0) = \Gamma^\dagger(H'_1) \oplus \Gamma^\dagger(H''_1). \tag{172}$$

and ultimately a splitting of the action of Γ on $\text{Ran } \Gamma^\dagger$ and the action of Γ^\dagger on $\text{Ran } \Gamma$:

$$\begin{aligned} \Gamma_R : U_2(H'_0) &\rightarrow U_1(H'_0), & \Gamma_R : U_2(H''_0) &\rightarrow U_1(H''_0); \\ \Gamma_R^\dagger : U_1(H'_0) &\rightarrow U_2(H'_0), & \Gamma_R^\dagger : U_1(H''_0) &\rightarrow U_2(H''_0). \end{aligned} \tag{173}$$

We now prove that H_2 is decomposed into the Ω_2 -orbits of $U_2(H'_0)$ and $U_2(H''_0)$. First, $\text{Ran } (\Gamma^\dagger \pi'_1) = U_2(H''_0)$, and since $\pi'_1 \Gamma e^{-i\Omega_2 t} \Gamma^\dagger \pi'_1 = 0$ for all t , we have $\pi'_1 \Gamma (\mathcal{O}_{\Omega_2}(U_2(H''_0))) = \{0\}$. It follows that $\mathcal{O}_{\Omega_2}(U_2(H''_0))$ is orthogonal to $U_2(H'_0)$. Setting

$$H'_2 = \mathcal{O}_{\Omega_2}(U_2(H'_0)) \quad \text{and} \quad H''_2 = \mathcal{O}_{\Omega_2}(U_2(H''_0)), \tag{174}$$

we have $H'_2 \perp H''_2$ and

$$\text{Ran } \Gamma^\dagger \subset H'_2 \oplus H''_2 \subset H_2. \tag{175}$$

As $H'_2 \oplus H''_2$ is an Ω_2 -invariant subspace of H_2 and $(\mathcal{H} = H_1 \oplus H_2, \Omega)$ is minimal as a conservative extension of $(H_1, \Omega_1, a_1(t))$, we obtain

$$H_2 = H'_2 \oplus H''_2. \tag{176}$$

By part (i) of Theorem 19, $(\mathcal{H}', \Omega \upharpoonright \mathcal{H}')$ is minimal as a conservative extension of $(H'_1, \Omega_1 \upharpoonright H'_1, a'_1(t))$. \square

The following statement describes a relation between the coupling operator Γ and s -invariant decompositions, for Γ with pure point spectrum as in Theorem 18.

Theorem 22. (*coupling operator and s -invariant decomposition*) *Suppose that Γ_0 has pure point spectrum.*

i. *Let*

$$H_1 = \bigoplus_{\alpha=1}^n H_{1\alpha} \quad \text{and} \quad H_2 = \bigoplus_{\alpha=1}^n H_{2\alpha} \tag{177}$$

be orthogonal decompositions such that $H_{1\alpha} \oplus H_{2\alpha}$ is invariant under Ω for each α , that is, the system (\mathcal{H}, Ω) splits as

$$\mathcal{H} = \bigoplus_{\alpha=1}^n (H_{1\alpha} \oplus H_{2\alpha}). \tag{178}$$

Then there exists a decomposition of Γ_0 into rank-one operators

$$\Gamma_0 = \sum_{q=1}^r \sqrt{\gamma_q} |g_{0q}\rangle \langle g_{0q}|, \quad \langle g_{0p} || g_{0q}\rangle = \langle g_{0p} || g_{0q}\rangle = \delta_{pq}, \tag{179}$$

giving rise to a decomposition of Γ into rank-one operators:

$$\Gamma = \sum_{q=1}^r \sqrt{\gamma_q} |g_q\rangle \langle g'_q|, \quad \langle g_p || g_q\rangle = \langle g'_p || g'_q\rangle = \delta_{pq}, \tag{180}$$

such that, for $q = 1, \dots, r$, there exists α such that $g_q \in H_{1\alpha}$ and $g'_q \in H_{2\alpha}$.

ii. *Conversely, assume that \mathcal{H} is reconstructible, and let a decomposition (180) of Γ be given arbitrarily. Let G be the graph with vertex set $\{1, \dots, r\}$ having an edge between p and q if and only if one of the following holds:*

$$\mathcal{O}_{\Omega_1}(g_p) \not\perp g_q, \quad \mathcal{O}_{\Omega_2}(g'_p) \not\perp g'_q. \tag{181}$$

Let V_1, \dots, V_n be the vertex sets of the connected components of G , and put

$$H_{1\alpha} = \mathcal{O}_{\Omega_1}(\{g_q : q \in V_\alpha\}), \quad H_{2\alpha} = \mathcal{O}_{\Omega_2}(\{g'_q : q \in V_\alpha\}). \tag{182}$$

(a) *$H_1 = \bigoplus_{\alpha=1}^n H_{1\alpha}$ and $H_2 = \bigoplus_{\alpha=1}^n H_{2\alpha}$ are orthogonal decompositions, and $H_{1\alpha} \oplus H_{2\alpha}$ is invariant under Ω for each α .*

- (b) $\mathcal{H} = \bigoplus_{\alpha=1}^n (H_{1\alpha} \oplus H_{2\alpha})$ is the finest s -invariant decomposition with the property that each g_q is in one of the $H_{1\alpha}$. This means that, if $\mathcal{H} = \bigoplus_{\beta=1}^m (H_1^\beta \oplus H_2^\beta)$ is another such decomposition, then, for all $\alpha = 1, \dots, n$, there is β such that $H_\alpha \subset H_i^\beta$, for $i = 1, 2$.
- iii. If all of the eigenspaces of $\Gamma\Gamma^\dagger$ are of dimension 1, then there exists a unique finest s -invariant decomposition of (\mathcal{H}, Ω) (see form (178)). This means that, if $\mathcal{H} = \bigoplus_{\beta=1}^m (H_1^\beta \oplus H_2^\beta)$ is any other s -invariant decomposition of (\mathcal{H}, Ω) , then for all $\alpha = 1, \dots, n$, there exists β such that $H_{i\alpha} \in H_i^\beta$.

Observe that the conditions (181) are equivalent to

$$\mathcal{O}_{\Omega_1}(g_p) \perp \mathcal{O}_{\Omega_1}(g_q), \quad \mathcal{O}_{\Omega_2}(g'_p) \perp \mathcal{O}_{\Omega_2}(g'_q). \tag{183}$$

Proof:

- i. From the Ω -invariance of $H_{1\alpha} \oplus H_{2\alpha}$ for each $\alpha = 1, \dots, n$, we infer that $\Gamma^\dagger(H_{1\alpha}) \subset H_{2\alpha}$ and $\Gamma(H_{2\alpha}) \subset H_{1\alpha}$ for each α . Define

$$\Gamma_\alpha := \Gamma \upharpoonright H_{2\alpha} : H_{2\alpha} \rightarrow H_{1\alpha}. \tag{184}$$

It is straightforward to verify that

$$\Gamma_\alpha^\dagger = (\Gamma_\alpha)^\dagger = \Gamma^\dagger \upharpoonright H_{1\alpha} : H_{1\alpha} \rightarrow H_{2\alpha}, \tag{185}$$

and we obtain a decomposition of Γ :

$$\Gamma = \sum_{\alpha=1}^n \Gamma_\alpha \pi_{2\alpha}, \tag{186}$$

where $\pi_{2\alpha}$ is the orthogonal projection to $H_{2\alpha}$. For each α , Γ_α admits a decomposition into rank-one operators

$$\Gamma_\alpha = \sum_{q=1}^{r_\alpha} \sqrt{\gamma_q} |g_{\alpha q}\rangle \langle g'_{\alpha q}|, \quad \langle g_{\alpha p} || g_{\alpha q}\rangle = \langle g'_{\alpha p} || g'_{\alpha q}\rangle = \delta_{pq}, \tag{187}$$

where $\gamma_{\alpha q} > 0$ are the eigenvalues of $\Gamma_\alpha \Gamma_\alpha^\dagger$ with corresponding orthonormal eigenvector basis $\{g_{\alpha q}\}_{q=1}^{r_\alpha}$ for $H_{1\alpha}$ and $\{g'_{\alpha q}\}_{q=1}^{r_\alpha}$ for $H_{2\alpha}$. Let $\{\gamma_q\}_{q=1}^r$ be an arrangement of $\{\gamma_{\alpha q} : q = 1, \dots, r_\alpha, \alpha = 1, \dots, n\}$ and $\{g_q\}_{q=1}^r$ and $\{g'_q\}_{q=1}^r$ the corresponding arrangements of the eigenvectors $\{g_{\alpha q}\}$

and $\{g'_\alpha q\}$. We obtain the required form

$$\Gamma = \sum_{\alpha=1}^n \Gamma_\alpha \pi_{2\alpha} = \sum_{q=1}^r \sqrt{\gamma_q} |g_q\rangle \langle g'_q|. \tag{188}$$

- ii. (a) Assume that $n > 1$, and let α and β be given with $1 \leq \alpha, \beta \leq n$ and $\alpha \neq \beta$. Let $p \in V_\alpha$ and $q \in V_\beta$ be given. Since there is no edge in G between p and q , we see that $\mathcal{O}_{\Omega_1}(g_p) \perp g_q$. $H_{1\alpha}$ is the smallest Ω_1 -invariant subspace of H_1 containing $\{g_p : p \in V_\alpha\}$, and is therefore equal to

$$H_{1\alpha} = \sum \{\mathcal{O}_{\Omega_1}(g_p) : p \in V_\alpha\}. \tag{189}$$

We infer that $g_q \in H_1 \ominus H_{1\alpha}$ for all $q \in V_\beta$. By the self-adjointness of Ω_1 , $H_1 \ominus H_{1\alpha}$ is Ω_1 -invariant, so that $H_{1\beta} = \mathcal{O}_{\Omega_1}(\{g_q : q \in V_\beta\}) \subset H_1 \ominus H_{1\alpha}$, and we conclude that $H_{1\alpha} \perp H_{1\beta}$. Now, $\mathcal{O}_{\Omega_1}(\text{Ran } \Gamma) = \mathcal{O}_{\Omega_1}(\{g_q\}_{q=1}^r) = \bigoplus_{\alpha=1}^n H_{1\alpha}$, and since (\mathcal{H}, Ω) is reconstructible, $H_1 = H_{1c} = \mathcal{O}_{\Omega_1}(\text{Ran } \Gamma)$ by Theorem 9. Therefore $H_1 = \bigoplus_{\alpha=1}^n H_{1\alpha}$. The analogous argument proves that $H_2 = \bigoplus_{\alpha=1}^n H_{2\alpha}$. We now prove the invariance of $H_{1\alpha} \oplus H_{2\alpha}$ under Ω . $H_{i\alpha}$ is by construction invariant under Ω_i . Let $v \in H_{1\alpha}$. Then $v = u + w$ for some $u \in \text{span } \{g_q : q \in V_\alpha\}$ and $w \in H_{1\alpha} \ominus \text{span } \{g_q : q \in V_\alpha\}$. Since $w \perp H_1 \ominus H_{1\alpha}$, we see that $w \perp \text{span } \{g_q : q \notin V_\alpha\}$, so that $w \perp \text{Ran } \Gamma$ and therefore $w \in \text{Ker } \Gamma^\dagger$. We now obtain $\Gamma^\dagger(v) = \Gamma^\dagger(u) \in \text{span } \{g'_q : q \in V_\alpha\} \subset H_{2\alpha}$. The invariance of $H_{1\alpha} \oplus H_{2\alpha}$ under Ω now follows.

- (b) Let $\mathcal{H} = \bigoplus_{\beta=1}^m (H_1^\beta \oplus H_2^\beta)$ be a Ω -invariant decomposition of \mathcal{H} such that each g_q is in one of the H_1^β . Fix α , and let $p, q \in V_\alpha$, so that $g_p, g_q \in H_{1\alpha}$ and $g'_q, g'_p \in H_{2\alpha}$. Since G contains an edge between p and q , one of the conditions (181) is satisfied. Assume that it is that $\Omega_{\Omega_1}(g_p) \not\perp g_q$ (the other case is handled analogously). Let β be such that $g_p \in H_1^\beta$; it follows that $g'_p \in H_2^\beta$. Since $\mathcal{O}_{\Omega_1}(g_p) \subset H_1^\beta$ and $H_1^\beta \perp H_1^{\beta'}$ for $\beta \neq \beta'$, we have $g_q \in H_1^\beta$. We conclude that $g_q \in H_1^\beta$ and $g'_q \in H_2^\beta$ for all $q \in V_\alpha$, so that $H_{1\alpha} \subset H_1^\beta$ and $H_{2\alpha} \subset H_2^\beta$.

- iii. Suppose the spectrum of Γ_0 is simple. Then the representations (179) and (179) are unique, so that by part (i), each decomposition $\bigoplus_{\beta=1}^m (H_1^\beta \oplus H_2^\beta)$ has the property that each g_q is in some H_1^β . The construction of the $H_{i\alpha}$ from part (2) has the property that for each α , there exists a β such that $H_{i\alpha} \subset H_i^\beta$ for $i = 1, 2$.

□

Remark 23. If we do not assume in part (2) that \mathcal{H} is reconstructible, then the result can be applied to the minimal reconstructible subsystem $H_{1c} \oplus H_{2c}$ of \mathcal{H} . The decoupled components H_{id} can be decomposed into Ω_i -invariant subspaces arbitrarily, and these are automatically Ω -invariant because they are contained in $\text{Ker } \Gamma^\dagger$ or $\text{Ker } \Gamma$.

The first half of Theorem 22 generalizes to operators with arbitrary spectrum. We remind the reader of the standard definition of an abstract resolution of the identity:

Definition 24. (*resolution of identity*) Given a set X and a σ -algebra \mathcal{B} of subsets of X containing the empty set and X , we say that $\pi : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ is a resolution of the identity $\mathbb{I}_{\mathcal{H}}$ if

- i. $\pi(A)$ is an orthogonal projection for all $A \in \mathcal{B}$,
- ii. $\pi(\emptyset) = 0, \pi(X) = \mathbb{I}_{\mathcal{H}}$,
- iii. $\pi(X \setminus A) = \mathbb{I}_{\mathcal{H}} - \pi(A)$ for all $A \in \mathcal{B}$,
- iv. $\pi(A_1 \cap A_2) = \pi(A_1)\pi(A_2)$ for all $A_1, A_2 \in \mathcal{B}$,
- v. $\pi(A_1 \cup A_2) = \pi(A_1) + \pi(A_2)$ for all $A_1, A_2 \in \mathcal{B}$ with $A_1 \cap A_2 = \emptyset$,
- vi. $\pi\left(\bigcap_{i=1}^{\infty} A_i\right) = s \lim_{i=1}^{\infty} \pi(A_i)$ for all sequences $\{A_i\}_{i=1}^{\infty}$ from \mathcal{B} with $A_{i+1} \subset A_i$.

π is said to commute with an operator T if, for all $A \in \mathcal{B}, \pi(A)T = T\pi(A)$.

These properties are not independent. For example, property (v) is implied by the first four, but we include it because of its conceptual relevance.

The statement below is partial generalization of Theorem 18 for Γ with general spectrum.

Theorem 25. Let π be a resolution of the identity on \mathcal{H} that commutes with Ω and P_1 . Then there exists a resolution E_π of the identity on H_0 that commutes with Γ_0 , such that, for all $A \in \mathcal{B}$,

$$U_i E_\pi(A) = \pi(A)U_i, \quad i = 1, 2, \tag{190}$$

from which it follows that

$$\text{Ran}(U_i E_\pi(A)) = \text{Ran}(\pi(A)) \cap H_{i\Gamma}, \quad i = 1, 2. \tag{191}$$

The proof of Theorem 25 is based on the following lemma.

Lemma 26. *Let $G_1 \oplus G_2$ be s -invariant, with $G_1 \subseteq H_1$ and $G_2 \subseteq H_2$. Then there exists Γ_0 -invariant subspace G_0 of $H_{0\Gamma}$ such that*

$$U_i \pi_{G_0} = \pi_{G_i} U_i, \quad i = 1, 2. \quad (192)$$

In fact,

$$G_0 = U_1^{-1}(\text{Ran } \Gamma \cap G_1) = U_2^{-1}(\text{Ran } \Gamma^\dagger \cap G_2). \quad (193)$$

Proof: From Theorem 19, part (i), we see that $\mathcal{G} := G_1 \oplus G_2$ is invariant under $\overset{\circ}{\Omega}$ and therefore also under the self-adjoint operator $\overset{\circ}{\Gamma}$; in other words, $\overset{\circ}{\Gamma}$ commutes with the orthogonal projection $\pi_{\mathcal{G}}$ onto \mathcal{G} in \mathcal{H} . Therefore,

$$(\text{Ran } \overset{\circ}{\Gamma}) \cap \mathcal{G} = \overset{\circ}{\Gamma}(\mathcal{G}) = \pi_{\mathcal{G}}(\text{Ran } \overset{\circ}{\Gamma}), \quad (194)$$

which is seen from the decomposition $\text{Ran } \overset{\circ}{\Gamma} = \overset{\circ}{\Gamma}(\mathcal{G}) \oplus \overset{\circ}{\Gamma}(\mathcal{H} \ominus \mathcal{G})$. From $\pi_{\mathcal{G}} = \pi_{G_1} \oplus \pi_{G_2}$ and the definition of $\overset{\circ}{\Gamma}$, we find that (194) admits the decomposition

$$(\text{Ran } \Gamma) \cap G_1 \oplus (\text{Ran } \Gamma^\dagger) \cap G_2 = \Gamma(G_2) \oplus \Gamma^\dagger(G_1) = \pi_{G_1}(\text{Ran } \Gamma) \oplus \pi_{G_2}(\text{Ran } \Gamma^\dagger). \quad (195)$$

In addition, $\overset{\circ}{\Gamma}^2(\mathcal{G}) = \overset{\circ}{\Gamma}(\mathcal{G})$, from which we obtain

$$\Gamma(\Gamma^\dagger(G_1)) = \Gamma(G_2) \quad \text{and} \quad \Gamma^\dagger(\Gamma(G_2)) = \Gamma^\dagger(G_1). \quad (196)$$

$\overset{\circ}{\Gamma}(\mathcal{G})$ is invariant under $\overset{\circ}{\Gamma}_R$, and therefore also under the unitary self-adjoint involution $\overset{\circ}{U}$ on $\text{Ran } \Gamma^\circ = \text{Ran } \Gamma \oplus \text{Ran } \Gamma^\dagger$

$$\overset{\circ}{U} := (\overset{\circ}{\Gamma}_R^2)^{-1/2} \overset{\circ}{\Gamma}_R, \quad (197)$$

in which we take the square root

$$\left(\overset{\circ}{\Gamma}_R^2\right)^{1/2} = \begin{bmatrix} (\Gamma_R \Gamma_R^\dagger)^{1/2} & 0 \\ 0 & (\Gamma_R^\dagger \Gamma_R)^{1/2} \end{bmatrix}. \quad (198)$$

Using $\overset{\circ}{\Gamma}(\mathcal{G}) = \Gamma(G_2) \oplus \Gamma^\dagger(G_1)$ and that

$$\overset{\circ}{U} = \begin{bmatrix} 0 & U^{-1} \\ U & 0 \end{bmatrix}, \quad (199)$$

we obtain $U\Gamma(G_2) \subseteq \Gamma^\dagger(G_1)$ and $U^{-1}\Gamma^\dagger(G_1) \subseteq \Gamma(G_2)$, so that $U\Gamma(G_2) = \Gamma^\dagger(G_1)$, and since $U = U_2 U_1^{-1}$, we may define

$$G_0 := U_1^{-1}\Gamma(G_2) = U_2^{-1}\Gamma^\dagger(G_1). \quad (200)$$

G_0 is Γ_0 -invariant because

$$\Gamma_0(G_0) = U_1^{-1}\Gamma U_2 G_0 = U_1^{-1}\Gamma \Gamma^\dagger(G_1) = U_1^{-1}\Gamma(G_2) = G_0. \quad (201)$$

Finally, since U_1 takes $H_{0\Gamma}$ isomorphically to $H_{1\Gamma}$, we have

$$U_1\pi_{G_0} = \pi_{\Gamma(G_2)}U_1, \tag{202}$$

in which the domain of $\pi_{\Gamma(G_2)}$ is $H_{1\Gamma}$, and from (195), we see that π_{G_1} coincides with $\pi_{\Gamma(G_2)}$ on $H_{1\Gamma}$ so that

$$U_1\pi_{G_0} = \pi_{G_1}U_1. \tag{203}$$

$U_2\pi_{G_0} = \pi_{G_2}U_2$ is obtained analogously. □

Proof of Theorem 25. We define a map $E_\pi : \mathcal{B} \rightarrow \mathcal{L}(H_0)$ and show it is a resolution of the identity with the desired property. Let $A \in \mathcal{B}$ be given, and set $G_1 = \text{Ran } P_1\pi(A)$ and $G_2 = \text{Ran } P_2\pi(A)$. We put

$$E_\pi(A) = \pi_{G_0}, \tag{204}$$

where G_0 is provided by Lemma 26; the property desired in the Theorem is thus provided by the lemma. Properties (i) and (ii) of a resolution of the identity are trivially verified for E_π . To see property (iii), let $\tilde{G}_0 = E_\pi(X \setminus A)$ and $\tilde{G}_i = H_i \ominus G_i = \text{Ran } P_i\pi(X \setminus A)$, and use

$$\pi_{G_0} = U_i^{-1}\pi_{G_i}U_i \quad \text{and} \quad \pi_{\tilde{G}_0} = U_i^{-1}\pi_{\tilde{G}_i}U_i, \tag{205}$$

to calculate

$$E_\pi(A) + E_\pi(X \setminus A) = \pi_{G_0} + \pi_{\tilde{G}_0} = U_i^{-1}(\pi_{G_i} + \pi_{\tilde{G}_i})U_i = U_i^{-1}\mathbb{I}_{H_i}U_i = \mathbb{I}_{H_0}. \tag{206}$$

To prove property 4 of Definition 24, let $A, B \in \mathcal{B}$, and compute

$$\begin{aligned} E_\pi(A \cap B) &= U_1^{-1}\pi(A \cap B)U_1 = U_1^{-1}\pi(A)\pi(B)U_1 \\ &= U_1^{-1}\pi(A)U_1U_1^{-1}\pi(B)U_1 = E_\pi(A)E_\pi(B). \end{aligned} \tag{207}$$

For property 5 we compute

$$\begin{aligned} \text{slim}_{n \rightarrow \infty} E_\pi(A_n) &= \text{slim}_{n \rightarrow \infty} U_1^{-1}\pi(A_n)U_1 = U_1^{-1} \text{slim}_{n \rightarrow \infty} (\pi(A_n)U_1) \\ &= U_i^{-1}\pi\left(\bigcap_{n=1}^{\infty} A_n\right)U_1 = E_\pi\left(\bigcap_{n=1}^{\infty} A_n\right). \end{aligned} \tag{208}$$

□

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